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VOLUME XVII.

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PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

SEVENTEENTH SESSION, 1898-99.

First Meeting, 11th November, 1898.

J. B. CLARK, Esq., M.A., F.R.S.E., President, in the Chair.

For this Session the following Office-bearers were elected:—

President - - Dr A. MORGAN, M.A., F.R.S.E.
Vice-President - Mr R. F. MUIRHEAD, M.A., B.Sc.
Hon. Secretary - Mr J. W. BUTTERS, M.A., B.Sc., F.R.S.E.
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Mr WM. A. LINDSAY, M.A. (*Convener*).
Dr C. G. KNOTT, F.R.S.E.
Mr CHAS. TWEEDIE, M.A., B.Sc., F.R.S.E.

Committee.

Messrs J. D. HAMILTON DICKSON, M.A., F.R.S.E.;
GEO. DUTHIE, M.A., F.R.S.E.; and JAMES MITCHELL, M.A., B.Sc.

Geometrical Theory of the Hyperbolic Functions.

By W. L. Thomson, M.A.

Figure 1.

1. If PQRS be an Hyperbola, OE, OF its asymptotes, P, Q, R, S any points on it such that the sectorial area OPQ = sectorial area ORS; and if PA, QB, RC, SD be ordinates to one asymptote and parallel to the other, it is known that

$$OA : OB = OC : OD, \quad \text{and} \quad PA : QB = RC : SD \quad (1).$$

Hence if A, B, C... be taken so that OA, OB, OC... are in continued proportion, the areas OPQ, OQR, ORS... are all equal, and since the number of points can be made as large as we please, the sum of the sectorial areas can be made as large as we please.
∴ The area between an asymptote, the curve, and any radius vector is infinite.

Figure 2.

2. Let P be any point on a rectangular hyperbola whose asymptotes are OE, OF and axis OA. Draw PM perpendicular to OA meeting OE in p, PB perpendicular to OE, Pp' parallel to OA and AD perpendicular to OE.

Then $\widehat{Op}M = 45° = \widehat{Bp'}P \quad \therefore \; Bp = BP = Bp'.$

From similar triangles

$$\frac{OB + BP}{OM} = \frac{OA}{OD} \quad (2) \qquad \text{and} \qquad \frac{OB - BP}{PM} = \frac{OA}{OD} \quad (3);$$

$$\therefore \; \frac{OB + BP}{OA} = \frac{OM}{OD} \quad (4) \qquad \text{and} \qquad \frac{OB - BP}{OA} = \frac{PM}{OD} \quad (5).$$

Hence if Q be any other point on the curve, QC, QN perpendicular to asymptote and axis,

$$\frac{OC + CQ}{OA} = \frac{ON}{OD} \quad (6) \qquad \text{and} \qquad \frac{OC - CQ}{OA} = \frac{QN}{OD} \quad (7).$$

∴ Multiplying (4) and (6)

$$\frac{OB \cdot OC + BP \cdot CQ + OC \cdot BP + OB \cdot CQ}{OA^2} = \frac{OM \cdot ON}{OD^2};$$

but $$OA^2 = 2OD^2;$$

$$\therefore \quad OB.OC + BP.CQ + OC.BP + OB.CQ = 2OM.ON.$$

Similarly from (5) and (7)

$$OB.OC + BP.CQ - OC.BP - OB.CQ = 2PM.QN.$$

$\therefore$ Adding and dividing by 2

$$OC.OB + BP.CQ = OM.ON + PM.QN. \quad - \quad - \quad (8)$$

In the same way by multiplying (4) by (7) and (5) by (6), and adding we get

$$OB.OC - BP.CQ = OM.QN + ON.PM. \quad - \quad - \quad (9)$$

FIGURE 3.

3. Let PM, OM be ordinate and abscissa of any point on the right hand branch of the rectangular hyperbola whose axis is OA.

Let $OA = a$ and area $OAP = U$, and let $\frac{2U}{a^2} = u$.

Then u is our variable and the definitions are

$$\sinh u = \frac{PM}{OA}, \quad \cosh u = \frac{OM}{OA}, \quad \tanh u = \frac{PM}{OM},$$

$$\coth u = \frac{OM}{PM}, \quad \operatorname{sech} u = \frac{OA}{OM}, \quad \operatorname{cosech} u = \frac{OA}{PM}.$$

4. The functions so defined are independent of the particular hyperbola we take, that is to say, given u, $\sinh u$, etc., are all determinate.

For all rectangular hyperbolas are similar figures and taking them with the same asymptotes, the centre is the centre of similarity. Then, drawing OPP′ cutting any two in P, P′, (Fig. 3) P, P′ are corresponding points.

Now if $$u = \frac{2OAP}{OA^2}, \text{ it also } = \frac{2OA'P'}{OA'^2},$$

since corresponding areas are as the squares of corresponding lines.

Also PM, P′M′ are corresponding lines being parallel.

$\therefore \frac{PM}{OA} = \frac{P'M'}{OA'}$ *i.e.* $\sinh u$ depends only on u.

Similarly for the other ratios.

5. From the definitions we have

$$\sinh u = \frac{1}{\operatorname{cosech} u}, \quad \cosh u = \frac{1}{\operatorname{sech} u}, \quad \tanh u = \frac{1}{\coth u} = \frac{\sinh u}{\cosh u}.$$

Also from the known property of the rectangular hyperbola that $OM^2 - PM^2 = OA^2$ we derive the three equations

$$\cosh^2 u - \sinh^2 u = 1, \quad \tanh^2 u + \operatorname{sech}^2 u = 1, \quad \coth^2 u - \operatorname{cosech}^2 u = 1.$$

6. The signs of lines are determined as in the circular functions, and u is + if OP rotates counterclockwise, − if clockwise.

Hence it is evident that $\sinh u$, $\tanh u$, $\coth u$, and $\operatorname{cosech} u$ are odd functions, while $\cosh u$ and $\operatorname{sech} u$ are even functions.

FIGURE 4.

7. ADDITION THEOREM.

Let $OAP = U$, $OPQ = U'$.

Make $OAR = U'$.

Draw QM, PL, RN perpendicular to the axis,

QB, PC, RE perpendicular to the asymptote.

Then if $u = \frac{2U}{a^2}$ and $v = \frac{2U'}{a^2}$, $\frac{QM}{OA} = \sinh(u+v)$, $\frac{OM}{OA} = \cosh(u+v)$.

Now $\frac{OB - BQ}{OA} = \frac{QM}{OD}$, $\therefore \frac{OB - BQ}{2OD} = \frac{QM}{OA}$, since $OA^2 = 2OD^2$.

$$\therefore \frac{OB \cdot OD - BQ \cdot OD}{2OD^2} = \frac{QM \cdot OA}{OA^2};$$

$$\therefore QM \cdot OA = OB \cdot OD - BQ \cdot OD$$
$$= OE \cdot OC - PC \cdot RE \quad \text{by (1)},$$
$$= OL \cdot RN + ON \cdot PL \quad \text{by (9)}.$$

$$\therefore \frac{QM}{OA} = \frac{OL}{OA} \cdot \frac{RN}{OA} + \frac{ON}{OA} \cdot \frac{PL}{OA};$$

i.e. $\sinh(u+v) = \sinh u \cosh v + \cosh u \sinh v$.

Again $$\frac{OB + BQ}{2OD} = \frac{OM}{OA} \quad \text{from (4)},$$

$$\therefore \frac{OB \cdot OD + BQ \cdot OD}{2OD^2} = \frac{OM \cdot OA}{OA^2};$$

$$\therefore \quad OM \,.\, OA = OB \,.\, OD + BQ \,.\, DO$$
$$= OL \,.\, ON + RN \,.\, PL \,.$$
$$\therefore \quad \frac{OM}{OA} = \frac{OL}{OA} \cdot \frac{ON}{OA} + \frac{RN}{OA} \cdot \frac{PL}{OA};$$

i.e. $\cosh(u+v) = \cosh u \ \cosh v + \sinh u \ \sinh v.$

Whence also $\sinh(u-v) = \sinh u \ \cosh v - \cosh u \ \sinh v,$

and $\cosh(u-v) = \cosh u \ \cosh v - \sinh u \ \sinh v.$

Whence also we may obtain formulæ for multiples and sub-multiples of u. Whence also we obtain in the usual way

$$\cosh nu = \cosh^n u + {}_nC_2 \ \cosh^{n-2} u \sinh^2 u + \ . \ . \ . \text{ etc.} \qquad (10)$$

8. Fundamental Inequalities.

Figure 3.

$$\sinh u = \frac{PM}{\overline{OA}} = \frac{PM \,.\, OA}{OA^2} = \frac{2\Delta OAP}{OA^2} > \frac{2 \text{ sectorial area } OAP}{OA^2}$$
$$> \quad u.$$

Also $\sinh^2 u = \cosh^2 u - 1.$

$\therefore \quad \sinh u < \cosh u.$

$$\tanh u = \frac{PM}{OM} = \frac{AT}{OA}, \quad \text{(if AT is the tangent at A),}$$
$$= \frac{AT \,.\, OA}{OA^2} = \frac{2\Delta OAT}{OA^2}$$
$$< \frac{2 \text{ sectorial area } OAP}{OA^2}$$
$$< \quad u.$$

$$\therefore \quad \tanh u < u < \sinh u < \cosh u \ - \ - \ - \ - \qquad (12)$$

$$\text{Also} \quad \sinh u = 2 \sinh \tfrac{u}{2} \cosh \tfrac{u}{2}$$
$$= 2 \tanh \tfrac{u}{2} / \operatorname{sech}^2 \tfrac{u}{2}$$
$$= \frac{2 \tanh \frac{u}{2}}{1 - \tanh^2 \frac{u}{2}}$$
$$< \frac{2 \cdot \frac{u}{2}}{1 - \left(\frac{u}{2}\right)^2} < \frac{u}{1 - \frac{u^2}{4}}, \quad \text{since } u > \tanh u. \qquad (13)$$

$$\cosh \tfrac{u}{2} = \frac{1 + \tanh^2 \frac{u}{2}}{1 - \tanh^2 \frac{u}{2}} < \frac{1 + \frac{u^2}{4}}{1 - \frac{u^2}{4}}. \ - \ - \ - \qquad (14)$$

9. LIMITS.

From (12) $\dfrac{1}{\cosh u} < \dfrac{u}{\sinh u} < 1, \quad \therefore \underset{u=0}{\text{Lt.}} \dfrac{u}{\sinh u} = 1.$

$$\therefore \text{ Also } \underset{u=0}{\text{Lt.}} \frac{u}{\tanh u} = 1.$$

Again $$\frac{u}{n} < \sinh \frac{u}{n} < \frac{\frac{u}{n}}{1 - \frac{u^2}{4n^2}}, \quad \text{by (12) and (13),}$$

$$\therefore \quad 1 < \frac{\sinh \frac{u}{n}}{\frac{u}{n}} < \frac{1}{1 - \frac{u_2}{4n_2}}.$$

$$\therefore \quad 1 < \left(\frac{\sinh \frac{u}{n}}{\frac{u}{n}}\right)^n < \left(1 - \frac{u^2}{4n^2}\right)^{-n}.$$

Now $$\underset{n=\infty}{\text{Lt.}} \left(1 - \frac{u^2}{4n^3}\right)^{-n} = \underset{n=\infty}{\text{Lt.}} \left\{\left(1 - \frac{u^2}{4n^2}\right)^{-\frac{4n^2}{u^2}}\right\}^{\frac{u^2}{4n}}$$

$$= e^0 = 1.$$

$$\therefore \underset{n=\infty}{\text{Lt.}} \left(\frac{\sinh \frac{u}{n}}{\frac{u}{n}}\right)^n = 1 \quad \text{- - - (15)}$$

Again $$1 < \cosh \frac{u}{n} < \frac{1 + \frac{u^2}{4n^2}}{1 - \frac{u^2}{4n^2}} \quad \text{- by (14)}$$

$$\therefore \quad 1 < \left(\cosh \frac{u}{n}\right)^n < \left(1 + \frac{u_2}{4n^2}\right)^n \left(1 - \frac{u^2}{4n^2}\right)^{-n}.$$

The Limit of the last expression

$$= \underset{n=\infty}{\text{Lt.}} \left\{\left(1 + \frac{u^2}{4n^2}\right)^{\frac{4n^2}{u^2}}\right\}^{\frac{u^2}{4n}} \times \underset{n=\infty}{\text{Lt.}} \left\{\left(1 - \frac{u^2}{4n^2}\right)^{-\frac{4n^2}{u^2}}\right\}^{\frac{u^2}{4n}}$$

$$= e^0 \times e^0 = 1.$$

$$\therefore \underset{n=\infty}{\text{Lt.}} \left(\cosh \frac{u}{n}\right)^n = 1 \quad \text{- - - (16)}$$

10. From these inequalities we can get the expansion of $\cosh u$ and $\sinh u$ in terms of u, by the same method as is used in the case of the circular functions, *e.g.* for $\cosh u$.

From (10) we have

$$\cosh nu = \cosh^n u + \frac{n(n-1)}{1.2}\cosh^{n-2}u \sinh^2 u + \ldots$$

$$= \cosh^n u \left\{1 + \frac{n(n-1)}{1.2}\tanh^2 u + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\tanh^4 u + \ldots\right\}.$$

From the way we got this n must be an integer and the series terminates, but we may take n as large as we please.

Writing u for nu we have

$$\cosh u = \cosh^n \tfrac{u}{n}\left\{1 + \frac{n(n-1)}{1.2}\tanh^2 \tfrac{u}{n} + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\tanh^4 \tfrac{u}{n} + \ldots\right\}$$

$$= \cosh^n \tfrac{u}{n}\left\{1 + \frac{1(1-\frac{1}{n})}{1.2}u^2\frac{\tanh^2 \frac{u}{n}}{(\frac{u}{n})^2} + \frac{1(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{3}{n})}{1.2.3.4}u^4\frac{\tanh^4 \frac{u}{n}}{(\frac{u}{n})^4}\ldots\right\}$$

$$= \cosh^n \tfrac{u}{n}\{1 + v_2 + v_4 \ldots + v_{2r} + \mathrm{R}\}, \quad r \text{ being a fixed finite number.}$$

Here $\mathrm{R} = \dfrac{1(1-\frac{1}{n})(1-\frac{2}{n}).(1-\frac{2r-1}{n})}{1.2.\qquad 2r}.u^{2r}\left(\dfrac{\tanh \frac{u}{n}}{\frac{u}{n}}\right)^{2r} + \ldots$ a terminating series,

$$< \frac{1(1-\frac{1}{n})(1-\frac{2}{n})\ldots(1-\frac{2r-1}{n})}{1.2.\qquad 2r}u^{2r} + \ldots \text{ a terminating series,}$$

$$< \frac{1}{\underline{|2r}}u^{2r} + \frac{1}{\underline{|2r+2}}u^{2r+2} + \ldots \text{ to a finite number of terms,}$$

since $(1-\frac{1}{n}), \ldots 1-\frac{2r-1}{n}$, are all positive,

$$< \frac{1}{\underline{|2r}}u^{2r} + \frac{1}{\underline{|2r+2}}u^{2r+2} + \ldots \textit{ ad infinitum}$$

$$< \frac{u^{2r}}{\underline{|2r}}\left\{1 + \frac{u^2}{(2r+1)(2r+2)} + \frac{u^4}{(2r+1)(2r+2)(2r+3)(2r+4)} + \ldots \begin{matrix}\textit{ad}\\ \textit{inf.}\end{matrix}\right\}$$

$$< \frac{u^{2r}}{\underline{|2r}}\left\{1 + \frac{u^2}{(2r+1)^2} + \frac{u^4}{(2r+1)^4} + \ldots \quad \textit{ad infinitum}\right\}$$

$$< \frac{u^{2r}}{\underline{|2r}} \cdot \frac{1}{1-\frac{u^2}{(2r+1)^2}} \cdot \quad - \quad - \quad \cdot \quad - \quad - \quad - \quad (17)$$

This is true for all values of n, n, of course, being $> 2r$.

Hence it is true when n is infinite.

But $$\underset{n=\infty}{\text{Lt.}}\ (\cosh \tfrac{u}{n})^n = 1$$

and $$\underset{n=\infty}{\text{Lt.}}\ v_{2m} = \frac{u^{2m}}{\lfloor 2m},\qquad m \text{ being finite.}$$

$\therefore$ Since r is finite

$$\cosh u = 1 + \frac{u^2}{2!} + \frac{u^4}{4!} + \quad + \frac{u^{2r}}{2r!} + R_{2r},$$

where R_{2r} is subject to condition (17).

Now $$\underset{r=\infty}{\text{Lt.}}\ \frac{u^{2r}}{\lfloor 2r} \cdot \frac{1}{1 - \frac{u^2}{(2r+1)^2}} = 0$$

$$\therefore \underset{r=\infty}{\text{Lt.}}\ R_{2r} = 0.$$

Hence we may write $\cosh u = 1 + \frac{u^2}{2!} + \frac{u^4}{4!} + \ldots$ *ad infinitum.*

In the same way we get

$$\sinh u = u + \frac{u^3}{3!} + \frac{u^5}{5!} + \ldots \quad \textit{ad infinitum.}$$

whence $$\cosh u = \frac{e^u + e^{-u}}{2},$$

and $$\sinh u = \frac{e^u - e^{-u}}{2},$$ the usual definitions.

Second Meeting, 9th December 1898.

ALEXANDER MORGAN, Esq., M.A., D.Sc., President, in the Chair.

Vorlesungen über Geschichte der Mathematik, von Moritz Cantor. Dritter (Schluss) Band.

Dritte Abtheilung, 1727-1758.

A REVIEW : with special reference to the *Analyst* Controversy.

By Professor GEORGE A. GIBSON.

With the issue of the Third Part of the Third Volume, Mr Cantor completes his History of Mathematics, in accordance with the plan he sketched out for himself when he undertook the work. That the labour involved in collecting material and in reducing it to shape would be great, Mr Cantor doubtless well knew; but in all probability his most liberal estimate of the demands likely to be made upon his energies has been far exceeded; in any case, one can readily understand the feelings of satisfaction with which he writes the preface to the concluding volume.

It hardly requires to be stated that this history is certain to remain for many years the standard work on the subject with which it deals; in completeness, in accuracy, in clearness of arrangement, it stands unrivalled, and for the period which it covers is bound to be a permanent work of reference. Even the year (1758) with which the history closes, though doubtless somewhat arbitrary, is yet more appropriate than appears at first sight. The most characteristic achievement in pure mathematics that has to be recorded during the period covered by the third volume (1668-1758) is doubtless the introduction of the Infinitesimal Calculus; but in spite of the great impetus that the work of Newton, Leibniz, and the Bernoullis gave to mathematical discovery, there was for many years after the publication of the first discoveries considerable uncertainty as to the philosophical basis on which the Calculus was to be built. The great variety and importance of the results to which the new method led pushed into the background the necessity for a thorough investigation of the fundamental principles of that method; when the method was seriously called in question, appeal was usually made to the principles so clearly expounded by Archimedes. Whether the Calculus

was to be defended from the Newtonian or the Leibnizian standpoint, it was the usual practice to try and prove its consistency with the ancient geometry. Sometimes, indeed, the correctness of the principle was deduced from the accuracy of the conclusions—more particularly in the case of infinite series; but obviously such a defence as this would break down the moment it was seriously examined by any competent logician.

Now, in the period to which the concluding part of Mr Cantor's third volume is devoted, the basis of the Infinitesimal Calculus was subjected to a most searching examination, with the result, it seems to me, of placing the Calculus on as secure a foundation as the ancient geometry. It is not altogether inappropriate, therefore, that Mr Cantor's work should close when such a definite result had been fairly gained. I do not suppose, however, that it was this consideration that influenced Mr Cantor in his determination of the year to which he should bring down his history, but it was rather the practical difficulty of reducing to order the enormous detail which almost seems to overwhelm the historian of the last hundred and fifty years.

Whatever be the reason for stopping at the particular year adopted, there can be no doubt that this history is a boon to mathematicians that can hardly be overestimated; and the fact that the first volume is already in a second edition must be to Mr Cantor a most gratifying evidence that his labours have been appreciated. There can, besides, be little doubt that his work has very much widened the circle of those who are interested in historical inquiries, and has given a great impetus to historical research.

In directing the attention of the Society to the concluding part of this history, I propose to adopt the same method that I followed on a previous occasion,* and confine myself chiefly to one definite issue, namely, the controversy called forth by the publication of Berkeley's *Analyst*. In taking this course I am influenced, partly by the consideration that criticism of isolated details is in itself less suited to the ends our Society has in view, but mainly because Mr Cantor's account of that controversy is very meagre, and not commensurate, as I think, with the place it occupies in the development of mathematical thought. From a passage on p. 718, it would seem that Mr Cantor has no first-hand acquaintance with the numerous

* *Proc. Edin. Math. Soc.*, Vol. XIV.

rejoinders which the *Analyst* called forth, except what may be found in Berkeley's *Works* and in Maclaurin's *Fluxions*; nor need one be much surprised at this ignorance of the writings of Philalethes Cantabrigiensis, since these are only to be found (so far as I am aware) in extremely rare journals. It is unfortunate, however, that the much more valuable contributions of Benjamin Robins were not known to Mr Cantor, as these would have shown that the state of mathematical learning in England in the years immediately subsequent to the death of Newton was by no means so low as the incompetence of Philalethes would seem to indicate. Robins gave a complete and masterly defence of fluxions, and, in particular, laid down in clear and unambiguous form the doctrine of limits as the basis of the Infinitesimal Calculus. The work of Robins is not so inaccessible as that of Philalethes, since all his important contributions are included in his *Mathematical Tracts*, published, under the editorship of James Wilson, M.D., at London in 1761. To Robins, more than to Maclaurin, I think, is due the credit of expounding in systematic and consistent form the fundamental conception of a limit, and of freeing Newton's statements from the ambiguities which gave plausibility to Berkeley's attack. As this controversy is of real importance in mathematical history, and as it seems to be very imperfectly known, I hope it may be of some interest to relate it with as much detail as the space at my disposal allows.

The facts of Berkeley's life are sufficiently well known to justify me in proceeding at once to state the origin of his famous essay, entitled—*The Analyst; or a Discourse addressed to an Infidel Mathematician* (London, 1734). *The Analyst* was addressed to Dr Halley, and begins with these words:—"Though I am a stranger to your person, yet am I not, sir, a stranger to the reputation you have acquired in that branch of learning which hath been your peculiar study; nor to the authority that you therefore assume in things foreign to your profession; nor to the abuse that you, and too many more of the like character, are known to make of such undue authority, to the misleading of unwary persons in matters of the highest concernment, and whereof your mathematical knowledge can by no means qualify you to be a competent judge." * This introductory sentence indicates what the rest of Berkeley's contributions to the controversy abundantly confirms—that his object in writing the *Discourse* was not so much to assail the

* Berkeley's *Works* (Fraser's Edition), Vol. III., 257–8.

conclusions reached by the new analysis, as to call in question the claim of mathematicians *as such* to be regarded as authorities in matters of religious faith. He does not charge mathematicians as a body with infidelity; but, seeing that mathematical truth is usually considered to be the most logically demonstrated of all truths, he tries to destroy the presumption that the opinions of mathematicians in matters of faith are likely to be more logical and trustworthy than those of divines, by contending that in the much-vaunted fluxional calculus there are mysteries that are accepted without question by mathematicians, but that are at the same time not capable of logical demonstration. He lays special stress on the fact that it is the logic and not the conclusions of the mathematicians that he assails; thus * "I beg leave to repeat and insist that I consider the geometrical analyst as a logician, *i.e.*, so far forth as he reasons and argues; and his mathematical conclusions, not in themselves, but in their premises; not as true or false, useful or insignificant, but as derived from such principles and by such inferences."

The charge, of which so much was made in the first reply of Philalethes, that Berkeley had accused mathematicians, as a body, of infidelity, is totally false and need not further be referred to.

There is, however, another point of view from which Berkeley's *Discourse* has been regarded that may be here briefly referred to. De Morgan † considers it to have been a publication involving the principle of Dr Whately's argument against the existence of Bonaparte. "The Analyst," he says, "is a tract which could not have been written except by a person who knew how to answer it. But it is singular that Berkeley, though he makes his fictitious character nearly as clear as afterwards did Whately, has generally been treated as a real opponent of fluxions. Let us hope that the arch Archbishop will fare better than the arch Bishop." I confess that I have some difficulty in adopting this opinion of De Morgan's, though there is more to be said in its favour than appears at a first view, and it would explain some points in the later developments of the controversy—as, for example, the absence of any reply to Robins. On the whole, however, I consider De Morgan's hypothesis more ingenious than sound.

Berkeley's first argument is directed against the conceiv-

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* *Works*, III., p. 270.

† *Phil. Mag.* (S. 4) 1852, IV, 329, note.

ability of fluxions. Quoting from the introduction to the *Quadratura Curvarum*, he says, "a method hath been found to determine quantities from the velocities of their generating motions. And such velocities are called fluxions; and the quantities generated are called flowing quantities. These fluxions are said to be nearly as the increments of the flowing quantities, generated in the least equal particles of time; and to be accurately in the first proportion of the nascent, or in the last of the evanescent increments. Sometimes, instead of velocities, the momentaneous increments or decrements of undetermined flowing quantities are considered, under the appellation of moments. By moments we are not to understand finite particles. These are said not to be moments, but quantities generated from moments, which last are only the nascent principles of finite quantities. It is said that the minutest errors are not to be neglected in mathematics; that the fluxions are celerities not proportional to the finite increments, though ever so small; but only to the moments or nascent increments, whereof the proportion alone, and not the magnitude, is considered. And of the aforesaid fluxions there be other fluxions, which fluxions of fluxions are called second fluxions. And the fluxions of these second fluxions are called third fluxions; and so on."

The description of *moments* in the above passage is not contained in the *Quadratura*, but is taken from the second Lemma of the second book of the *Principia*; this point should be specially noted, as one object of the introduction to the *Quadratura* is to show that the doctrine of fluxions is independent of infinitesimals, and the use of moments *there* would have endangered the contention. Whether Newton's exposition is really independent of the principle of moments is, of course, a different question.

Now Berkeley objects that the imagination is very much strained and puzzled to frame clear ideas of the least particles of time, or the least increments generated therein; and much more so to comprehend the moments, or those increments of the flowing quantities in *statu nascenti*, in their very first origin or beginning to exist, before they become finite particles. "And it seems much more difficult to conceive the abstracted velocities of such nascent imperfect entities. But the velocities of the velocities—the second, third, fourth, fifth velocities, &c.—exceed, if I mistake not, all human understanding. Certainly, in any sense, a second or third fluxion seems an obscure mystery. The incipient celerity of an incipient celerity, the nascent augment

of a nascent augment, *i.e.*, of a thing which hath no magnitude —take it in what light you please, the clear conception of it will, if I mistake not, be found impossible; whether it be so or not, I appeal to the trial of every thinking reader."

Berkeley then makes similar observations on the differentials of the foreign mathematicians, and closes this part of his argument with the assertion that if we penetrate beneath the symbols we shall discover much emptiness, darkness, and confusion, even direct impossibilities and contradictions.

Berkeley's contention here is practically that it is nonsense to speak of velocities as being proportional to moments, since moments are not finite quantities, and therefore a ratio of moments is a ratio of things that have no magnitude. Or, in different language, if the increment of space δx be generated in time δt, then δx and δt are not moments so long as they are finite; the fluxion of x is the ratio of δx to δt when these are no longer finite; but if they are not finite, then there are no magnitudes which can bear a ratio to each other. Hence even a first fluxion, as thus defined, has no real meaning.

It goes without saying that if Berkeley has correctly interpreted Newton, his criticism is just; but that criticism rests on a false interpretation of Newton's conception of prime and ultimate ratios, as will be seen later.

Berkeley then examines the proof for finding the fluxion of a rectangle AB given in the *Principia*, Book II., Lemma 2. When the sides A, B are deficient by half their moments $\frac{1}{2}a$, $\frac{1}{2}b$ respectively, the rectangle is

$$\mathrm{AB} - \tfrac{1}{2}b\mathrm{A} - \tfrac{1}{2}a\mathrm{B} + \tfrac{1}{4}ab;$$

when the sides are greater by half their moments the rectangle is

$$\mathrm{AB} + \tfrac{1}{2}b\mathrm{A} + \tfrac{1}{2}a\mathrm{B} + \tfrac{1}{4}ab.$$

If the former rectangle be subtracted from the latter, there remains $b\mathrm{A} + a\mathrm{B}$, and this is the increment of AB generated by the entire increments a, b of the sides.

The objection is urged that the increment is really

$$(\mathrm{A}+a)(\mathrm{B}+b) - \mathrm{AB}, \textit{ i.e., } b\mathrm{A} + a\mathrm{B} + ab,$$

and therefore the term ab has been illegitimately neglected, contrary to Newton's own principles that even the smallest errors are not to be neglected in mathematics. To emphasise the difficulty of getting rid of the term ab by legitimate methods, Berkeley next examines the proof in the *Quadratura* of the fluxion of x^n. His criticism is based on the following lemma :—

"If, with a view to demonstrate any proposition, a certain point is supposed, by virtue of which certain other points are attained; and such supposed point be itself afterwards destroyed or rejected by a contrary supposition; in that case, all the other points attained thereby, and consequent thereupon, must also be destroyed and rejected, so as from thenceforward to be no more supposed or applied in the demonstration."

To find the fluxion of x^n, suppose x by flowing to become $x+o$, then x^n will become

$$x^n + nox^{n-1} + \frac{n(n-1)}{2}o^2x^{n-2} + \text{ etc.}$$

The increments of x and of x^n are therefore to each other

$$\text{as } o \text{ to } nox^{n-1} + \frac{n(n-1)}{2}o^2x^{n-2} + \text{ etc.},$$

$$\text{or as 1 to } nx^{n-1} + \frac{n(n-1)}{2}ox^{n-2} + \text{ etc.}$$

Let now the increments vanish, and their last proportion is 1 to nx^{n-1}, so that the fluxion of x is to that of x^n as 1 to nx^{n-1}.

Berkeley objects that this reasoning is not fair or conclusive because the supposition that the increments vanish, destroys the former supposition that there were increments, and therefore, we must suppose that everything derived from the supposition of their existence should vanish with them. He dilates at considerable length on the want of logic shown in this reasoning, and then proceeds to show that when conclusions, known otherwise to be sound, are obtained by this method, the logic is faulty all the same, as the right conclusion is reached by compensation of errors. The examples he takes are drawn from the finding of tangents by the use of differentials, and his criticism at this part is aimed rather at Leibniz than Newton. These examples are reproduced by Mr Cantor, and I will pass them over with the remark that they are of more than usual interest in view of subsequent attempts to establish the calculus on the principle of compensation.

The *Analyst* is a lengthy pamphlet and contains much acute criticism of mathematical methods; but its contentions are in essence these two:—(1) the conception of fluxions is unintelligible, inasmuch as they suppose the ratio of quantities that have no magnitude, for, in Berkeley's view, prime and ultimate ratios are such; (2) the demonstration of the value of a fluxion, say that of x^n, rests on the violation of an axiomatic canon of sound reasoning.

As may be supposed, such an uncompromising assault on the

fundamental principles of the new analysis created great excitement in the mathematical world; not only was Berkeley accused of charging mathematicians as a body with infidelity, but he had cast a slur on the genius of Newton, who had emerged, as was universally believed in England, triumphant at every point from the great dispute about the invention of the calculus. For many years every work on fluxions that appeared in England tried, with more or less success, to demolish Berkeley's arguments; the first attempt in that direction, as was right and fitting, came from Cambridge, in the shape of a pamphlet bearing the title *Geometry, no Friend to Infidelity; or, A Defence of Sir Isaac Newton and the British Mathematicians, in a Letter addressed to the Author of the Analyst. . . . By Philalethes Cantabrigiensis.* The pamphlet was published at London in 1734, the letter being dated, "Cambridge, April 10, 1734."

Mr Cantor (p. 718) attributes the authorship to two Cambridge professors, Conyers Middleton and Robert Smith; Philalethes asserts that there was but one author, though he admits consulting Smith on various points. There is little doubt, however, that Philalethes was not Middleton, but Dr Jurin; Middleton's controversies with Berkeley had reference to other subjects than fluxions.

Philalethes reduces the objections of the *Analyst* to three, namely: (1) The obscurity of the doctrine of fluxions; (2) False reasoning in it used by Sir Isaac Newton and implicitly received by his followers; (3) Artifices and Fallacies used by Newton to make this false reasoning pass upon his followers.

In regard to (1) Philalethes admits that the doctrine is not without its difficulties, but denies that these difficulties are insuperable for any "thinking reader" who is well versed in geometry; the appeal to thinking readers who are not so versed is inadmissible.

The essence of the defence is made under head (2), and an extremely weak defence it is. To account for the neglecting of the term $a\ b$ in finding the increment of the rectangle A B, he contends in the first place that mathematicians can very well estimate the effect of omitting that term, and they know that the error is not so much as a pin's head compared with a globe the size of the earth, or of the sun, or even of the orb of the fixed stars. At a later stage of the controversy, indeed, he somewhat resiles from this position by saying that he had introduced this comparison by way of popular exposition. In the second place, however, he maintains that "rigorously speaking, the

moment of the rectangle AB is not, as you suppose, the increment of the rectangle AB; but it is the increment of the rectangle $(A - \frac{1}{2}a)(B - \frac{1}{2}b)$."

To clear up the point he observes that the word moment is used, both by Newton and by Berkeley, to signify indifferently either an increment or a decrement, that the increment of AB is, on Berkeley's showing, $aB + bA + ab$ and that the decrement should, therefore, be $aB + bA - ab$. He then asks which of these expressions is to be considered the moment, and maintains that the moment is the arithmetic mean of the two. This extraordinary contention about the arithmetic mean is apparently a favourite with Philalethes, as he repeats it in his second reply to Berkeley; afterwards, under stress of Robins' criticism, he alleges that the argument was intended to be taken not as rigorous demonstration, but only as *against Berkeley*. As a matter of fact, however, he tries to patch it up in later articles, where he is opposing not Berkeley but Robins—with disastrous results to the doctrine he is defending.

A curious illustration of the blindness of Philalethes to Newton's doctrine, as expounded in the introduction to the *Quadratura* (*volui ostendere quod in Methodo Fluxionum non opus sit Figuras infinite parvas in Geometriam introducere*), occurs in the closing sentence of this section:—"Lastly, to remove all scruple and difficulty about this affair, I must observe that the moment of the rectangle AB determined by Sir Isaac Newton, namely $aB + bA$, and the increment of the same rectangle determined by yourself, namely, $aB + bA + ab$, are perfectly and exactly equal, supposing a and b to be diminished *ad infinitum*, and this by the lemma just now quoted." (*Principia*, Book I., Sect. 1, Lemma 1).

Philalethes then turns to the criticism of the method of finding the fluxion of x^n. He objects to Berkeley's translation of the phrase "*evanescent jam augmenta illa et eorum ratio ultima erit*," namely, "let the increments be nothing," or "let there be no increments." He says "Ought it not to be thus translated:—'Let the augments now become evanescent, let them be upon the point of evanescence.' . . . . Do not the words *ratio ultima* stare us in the face, and plainly tell us that there is a last proportion of evanescent increments, yet there can be no proportion of increments which are nothing, of increments which do not exist!" What precisely he means by the *last* proportion of evanescent increments may be left out of account at present, as this part of the controversy is fully dealt with at a later stage.

Of the third main objection (artifices and fallacies, etc.), little can be said, as Philalethes, in that part of his reply, gives proof of nothing except his command of strong language.

He then goes on to consider the charge of reaching true conclusions by the compensation of errors. His defence, however, is a mere repetition of the argument for the equality of

$$a\mathrm{B}+b\mathrm{A} \quad \text{and} \quad a\mathrm{B}+b\mathrm{A}+ab.$$

There can be little doubt that Philalethes was in blank ignorance of the characteristic features of Newton's doctrine, as expounded, for example, in the introduction to the *Quadratura*. No doubt, the language of Newton was not free from ambiguity, and there is as little doubt that in his first attempts to formulate a doctrine of fluxions, in the *De Analysi per aequationes* for example, he used infinitely little quantities in the sense of Cavallerius and the Leibnizian school. But as he himself expressly says, he used the method of prime and ultimate ratios to avoid the introduction into geometry of infinitely little quantities. The contentions of Philalethes about the equality of $a\mathrm{B}+b\mathrm{A}$ and $a\mathrm{B}+b\mathrm{A}+ab$ were enough to make Newton turn in his grave, and his whole manner of treating Berkeley's criticisms was in fact a powerful argument in their favour.

In 1735 Berkeley published *A Defence of Free-thinking in Mathematics, in answer to a Pamphlet of Philalethes Cantabrigiensis . . . Also an Appendix concerning Mr Walton's Vindication of the Principles of Fluxions against the Objections contained in the Analyst.*

Walton's *Vindication*, as well as his other contributions to the controversy, I have not seen; but if one may judge from this *Appendix* and from another tract of Berkeley's, entitled *Reasons for not replying to Mr Walton's Full Answer*, Walton was even less qualified than Jurin to demolish the arguments of *The Analyst*.

In the *Defence* Berkeley takes Philalethes to task for misrepresenting the religious side of the *Analyst*, asserts his right to expose the errors even of the greatest men, admits the great genius of the inventor of fluxions, but somewhat scornfully advises Philalethes to worship truth rather than Newton. He repeats his contention that a fluxion is incomprehensible, and tries so to unveil this mystery as that every reader of ordinary sense and reflection, not the mathematician merely, may be a competent judge of the points in dispute ; he charges Philalethes with apparent ignorance of Newton's attitude towards infinitely small quantities in the method he adopts to get rid of the rect-

angle *ab*, and directly traverses the interpretation Philalethes gives of the moment of the rectangle AB. He supports his translation of the phrase *evanescant jam augmenta illa* by stating that the expression

$$nx^{n-1} + \frac{n(n-1)}{2}ox^{n-2} + \text{ etc.},$$

can never be brought to nx^{n-1} except by supposing the increment *o* actually nothing, and further justifies his translation by quoting from the *De Analysi per aequationes*, where *evanescere* is rendered *esse nihil*. He strengthens his allusion to the "various arts and devices used by the great author of the fluxionary method" by describing the various accounts of his momentums and fluxions as inconsistent, and asks Philalethes to tell him "whether Sir Isaac's momentum be a finite quantity, or an infinitesimal, or a mere limit. If you say a finite quantity: be pleased to reconcile this with what he saith in the scholium of the second lemma of the first section of the first book of his Principles:—*Cave intelligas quantitates magnitudine determinatas sed cogita semper diminuendas sine limite.* If you say an infinitesimal: reconcile this with what is said in his introduction to the Quadratures:—*Volui ostendere quod in methodo fluxionum non opus sit figuras infinite parvas in geometriam inducere.* If you should say it is a mere limit: be pleased to reconcile this with what we find in the first case of the second lemma in the second book of the Principles:—*Ubi de lateribus A et B deerant momentorum dimidia, etc.*—where the moments are supposed to be divided." The assertion of Philalethes that the objection based on the compensation of errors had been foreseen and removed by Newton in the *Principia, Book I., Sect.* 1., is characterised as an unquestionable proof of the matchless contempt which Philalethes has for truth. After referring to the use of infinitesimals by the Marquis de l'Hopital, he insists that the question of errors in professed approximations is quite distinct from that of logical errors in the reasoning—a distinction which Philalethes is apparently unable to grasp—and then quotes the diverging conceptions of the principles of the modern analysis held by mathematicians with whom he had conversed.

Under the date "Cambridge, June 13, 1735," Philalethes replies to the *Defence* in a pamphlet entitled *The Minute Mathematician: or the Freethinker no Just Thinker.* After some pages of general invective, Philalethes takes up the challenge, to show that the principles of fluxions may be clearly

conceived. In several respects this second pamphlet is an improvement on the first, but in spite of occasional glimpses of Newton's real position, Philalethes only confirms the idea that, so far as he had definite conceptions, he stood on quite different ground from Newton, whom he was understood to be defending. Thus he says "A nascent increment is an increment just beginning to exist from nothing, or just beginning to be generated, but not yet arrived at any assignable magnitude how small soever." Again, he is in the regions of the infinitesimals, which Newton had condemned, when he asserts that "the magnitude of a moment is nothing fixed or determinate, is a quantity perpetually fleeting and altering till it vanishes into nothing; in short, that it is utterly unassignable." Or again—"What he (Newton) says, and what I contend for, is this. Though so long as a and b are real quantities their rectangle ab is a real quantity, and there is a real difference between the two quantities $a\mathrm{B}+b\mathrm{A}$ and $a\mathrm{B}+b\mathrm{A}+ab$; yet when by a continual diminution *ad infinitum* a and b vanish, their rectangle ab, or the difference between the two quantities $a\mathrm{B}+b\mathrm{A}$ and $a\mathrm{B}+b\mathrm{A}+ab$, vanishes likewise, and there is no longer any difference between those quantities, *i.e.*, those quantities are equal."

It is amazing that Philalethes did not see that this argument only proves, if it proves anything, that two magnitudes which are each nothing are equal; he simply confirms Berkeley's contention (*Defence*, sect. 32) that "for a fluxionist writing about momentums to argue that quantities must be equal because they have no assignable difference seems the most injudicious step that could be taken. For, it will thence follow that all homogeneous momentums are equal, and consequently the velocities, mutations, or fluxions, proportional thereto, are all likewise equal."

The contention of Philalethes is professedly based on the doctrine of prime and ultimate ratios expounded in Sect. 1, Book I. of the *Principia*, but he utterly misconceives that doctrine, interpreting it, it seems to me, exactly as Berkeley had done, so that an ultimate ratio is not the *limit* of a varying ratio, but the *last value* of a ratio. Berkeley very properly maintains that there is no last value of the augments except zero, so that the phrase "the ratio with which they vanish," used by Newton himself, and so often repeated by his expounders, does not represent any mathematical operation, and so far from explaining anything, requires explanation. No doubt the ordinary notions of what is meant by the value at a given

instant of a varying velocity furnish a starting point for the measure of a fluxion and of the ratios of fluxions, and it is on these notions that Maclaurin bases his exposition in the first volume of his *Fluxions*, the measure of a varying velocity as a limit, being with him a *theorem* rather than a definition (*Fluxions*, vol. I., sect. 67). But a reference to the measure of a varying velocity does not clear up the logical difficulty of the mathematical procedure for finding a fluxion as urged by Berkeley in the cases of the rectangle AB and of x^n. Maclaurin himself seems to recognise this when he comes to treat of *The Computations in the Method of Fluxions*, for he says (*Fluxions* Sect. 702) in the case where A increases at a constant rate by equal differences but B increases by differences that are always varying, "it is not so obvious how, the fluxion of A being supposed equal to its increment A, the variable fluxion of B is to be determined."

Newton's method is the thoroughly sound one of limits, and Berkeley's criticism was really based on a misinterpretation of Newton's terminology. For this misinterpretation there was some excuse on his part; Newton, as Robins conclusively proved, used at first the methods and the language of indivisibles, but after discovering the method of prime and ultimate ratios he discarded that of indivisibles, though he often used language borrowed from the older writings. The terminology of first and last ratios was unfortunate, as it lent itself too readily to an interpretation in the sense of indivisibles; and it was this interpretation that Berkeley and Philalethes alike proceeded upon. Were that interpretation correct, then Berkeley's contentions would in the main be fully justified.

To the second pamphlet of Philalethes, Berkeley made no reply, so that his disputes about fluxions are limited to the three tracts, *The Analyst*, *The Defence*, and *The Reasons for not replying to Mr Walton's full answer* with the *Appendix*. It would be interesting to know why he offered no criticism on the next publication that the controversy produced, namely, *A Discourse concerning the Nature and Certainty of Sir Isaac Newton's Methods of Fluxions and of Prime and Ultimate Ratios. By Benjamin Robins, F.R.S. (London, 1735)*. Whatever the reason might be, Berkeley now retired from the contest, and Robins and Philalethes began that long struggle in which Robins proved his immense superiority to his antagonist, alike in temper, in general mathematical learning, and in special knowledge of Newton's fluxionary methods.

Robins, born at Bath, in 1707, of a poor Quaker family, was

mainly self-taught in mathematics, and for some years supported himself in London as a teacher of mathematics. He is now best known by his great work, *New Principles of Gunnery*. In 1749 he was appointed Engineer-in-General to the East India Company, but his first undertaking, the planning of the defences of Madras, was no sooner accomplished than he was seized with a fever. Though he recovered, he never regained good health, and he died at Fort St David, July 29, 1751.

The *Discourse* is a masterpiece of its kind; in knowledge of the ancient geometry, in grasp of fundamental principles, in strength of logic, and in facility of expression, it is a splendid testimony not only to the intellectual power of its author, but to the profound logical difference between the method of indivisibles and the method of prime and ultimate ratios. As regards the interpretation of Newton's language, it should be borne in mind that Robins, as he more than once remarks in the course of the dispute with Philalethes, reached his conclusions through study of the ancient geometers and of Newton's writings, and he was thus thoroughly fitted to give the right interpretation of phrases that were in themselves ambiguous, or that were used in a different sense by the writers on indivisibles. This remark is of some importance, as Philalethes harped perpetually on a literal acceptation of words, and apparently could not or would not allow himself to see that his literal interpretation made sheer nonsense of Newton's doctrine.

Robins distinguishes between the doctrine of fluxions and that of prime and ultimate ratios. To avoid the imperfections of the method of indivisibles, Newton, he says, considered magnitudes as generated by a continued motion, and discovered a method to compare together the velocities wherewith homogeneous magnitudes increase; on the other hand, to facilitate the demonstrations, he invented the method of prime and ultimate ratios. The foundation of the method of fluxions is this principle, that "if the proportion between the celerity of increase of two magnitudes produced together is in all parts known, then the relations between the magnitudes themselves must from thence be discoverable." It is by means of proportions only that fluxions are applied to geometrical uses, for no determinate degree of velocity ever requires to be assigned for the fluxion of any one fluent.

Robins first proves by the method of exhaustions that the fluxion of x^n/a^{n-1} is to that of x as nx^{n-1}/a^{n-1} is to 1, and also finds the fluxion of a rectangle. The proofs, as might be expected, are somewhat long, but they possess all the evidence

of an Archimedean demonstration; beyond that there was no appeal. He afterwards gives Newton's own proof, by the method of prime and ultimate ratios, stating at this point that it is equally just with that by the method of exhaustions.

He next goes on to show how fluxions are to be applied to the drawing of tangents to curve lines, and to the mensuration of curvilinear spaces, the proofs being by the method of exhaustions, and then gives a very clear description of the higher orders of fluxions, the manner of determining them, and some of the uses to which they may be applied.

In passing to the consideration of prime and ultimate ratios, he gives a short account of the ancient method of measuring curvilinear spaces, shows that the area of the circle may be determined in the Archimedean manner by the use of one polygon only, and points out that Newton instituted upon this principle (of comparison with one polygon only) a briefer method of conception and expression for demonstrating this sort of propositions than was used by the ancients. He allows that "the concise form, into which Sir Isaac Newton has cast his demonstrations, may very possibly create a difficulty of apprehension in the minds of some unexercised in these subjects," but not of those who are versed in "the ancients," while the method is just and free from any defect in itself. (Sects. 89—94),

He states his interpretation of the first Lemma of Book I. of the *Principia* thus:—"In this method any fixed quantity, which some varying quantity, by a continual augmentation or diminution, shall perpetually approach, but never pass, is considered as the quantity, to which the varying quantity will at last or ultimately become equal; provided the varying quantity can be made in its approach to the other to differ from it by less than by any quantity how minute soever, that can be assigned" (Sect. 95). Again, "Ratios also may so vary, as to be confined after the same manner to some determined limit, and such limit of any ratio is here considered as that with which the varying ratio will ultimately coincide. From any ratio's having such a limit, it does not follow, that the variable quantities exhibiting that ratio have any final magnitude, or even limit, which they cannot pass" (Sects. 98, 99).

Of course, this view of the famous lemma at once demolishes great part of Berkeley's contentions. Robins is particularly emphatic in confuting the notion that the last ratio of two varying quantities is necessarily a ratio which the quantities themselves can ever bear to each other, and repeats the illus-

tration given by Newton at the end of the Scholium (*Prin., Bk. I., Sect.* 1), besides giving others in which the limit of each varying quantity is zero. He spends some time in explaining the meaning to be attached to the Latin words *evanescens* and *nascens*, and certainly his explanation is the most reasonable one, the only one, indeed, that is consistent with the general sense of Newton.

After his explanation of the conception upon which the doctrine is built, he proceeds "to draw out its first principles into a more diffusive form"; all that need be said is that his exposition is extremely clear, and is better than we often find in works of the present day. His explanation is followed by a statement and defence of Newton's own method of finding fluxions.

The *conclusion* of the *Discourse* is devoted to an explanation of the term *momentum*—a term not previously used in the tract. His explanation is that "in determining the ultimate ratios between the contemporaneous differences of quantities, it is often previously required to consider each of these differences apart, in order to discover how much of these differences is necessary for expressing that ultimate ratio. In this case Sir Isaac Newton distinguishes by the name of momentum so much of any difference as constitutes the term used in expressing this ultimate ratio." (Sect. 154). [It may be noted in passing that this separation of a difference into two parts is a favourite proceeding of Maclaurin.] Thus if o be the momentum of x, $nx^{n-1}o$ is the momentum of x^n; and $a\mathrm{B}+b\mathrm{A}$, not the whole increment $a\mathrm{B}+b\mathrm{A}+ab$, is called the momentum of AB, because so much only of the increment is required for determining the ultimate ratio of the increment of AB to the increment of MA, where M is any constant. He adds—"It must always be remembered that the only use which ought ever to be made of these momenta is to compare them with one another, and for no other purpose than to determine the ultimate or prime proportion between the several increments or decrements from whence they are deduced. Herein the method of prime and ultimate ratios essentially differs from that of indivisibles; for, in the method of indivisibles, momenta are considered absolutely as parts whereof their respective quantities are actually composed. But these momenta have no final magnitude which would be necessary to make them parts capable of compounding a whole by accumulation; yet their ultimate ratios are as truly assignable as the ratios between any quantities whatever." He justifies his explanation by the quotation

from the *Principia*, Book II., Lem. 2 :—*neque spectatur magnitudo momentorum, sed prima nascentium proportio.*

As he not only admits but contends in subsequent papers that Newton did not always avoid the method of indivisibles, it may be well to add that Robins offers the above explanation as one that "shall agree to the general sense of his (Newton's) description." In spite of his great admiration for Newton, he saw quite as clearly as Berkeley the absurdity of the exposition of Philalethes, and claimed that Newton himself explicitly allowed the defects of his earlier work and the possibility of interpreting his language in the sense of indivisibles.

The *Discourse* was not ostensibly directed against Berkeley, as Robins was averse to entering into controversy with Philalethes; he thought the best method of elucidating the truth was to make no reference to either disputant, but to re-state the Newtonian doctrine. He soon found, however, that direct conflict with Philalethes was impossible. In the number for October 1735 of a magazine published in London, and called *The Present State of the Republic of Letters*, Robins gave an account of the *Discourse*, and expressed himself more freely than in the tract regarding the possible misconceptions of Newton's position through Newton's own injudicious concessions in the matter of language to the prevalent method of indivisibles. He also explicitly discussed Newton's early use of that method of demonstration and the difficulty involved in his use of *momenta*.

The November number of the same magazine brought out an article by Philalethes, entitled *Considerations upon some Passages contained in two Letters to the Author of the Analyst, written in Defence of Sir Isaac Newton and the British Mathematicians*. After a moderately-worded introduction, he reduces Robins's objections to his representation of Newton's doctrines to these three :—

I. His explication of *Principia*, Book I., Lemma 1.

II. The sense of the Scholium to Book I., Sect. 1, particularly as to (1) the doctrine of Limits, (2) the meaning of the term evanescent or vanishing.

III. The sense of *Principia*, Book II., Lemma 2.

As to I., Philalethes maintains that Newton means not that the quantities or ratios are merely to be *considered* as ultimately equal, but that they do at last become "actually, perfectly, and absolutely equal." Thus the inscribed and circumscribed parallelograms of *Principia*, Book I., Sect. 1, Lemma 2, do ultimately

each coincide with the curvilinear figure which is the limit at which they arrive. In the same way in regard to II. (1) he holds that it is necessary that the variable quantity should *actually reach* the *limit*. The discussion of objection III. is much more guarded than in the corresponding passages of his criticism of Berkeley. He allows that the course taken by Newton (in his interpretation naturally) to find the difference of variable quantities is not rigorously geometrical in the case of higher products than two, of ABC for example, yet that it approaches nearer to geometric rigour than the method used by Leibniz. Incidentally he denies that Newton ever admitted of indivisibles.

Robins published in the December number of the *Republic* a *Review of some of the principal Objections, &c., with some Remarks on the different Methods that have been taken to obviate them*. He states very clearly, as I think, the view that Philalethes takes of Newton's expression *Fluxiones sunt in ultima ratione decrementorum evanescentium vel prima nascentium*, when he says that that explanation "endeavours to show how this imagined difficulty (of a ratio between nascent or evanescent magnitudes) may be avoided, not by considering these evanescent decrements and nascent augments as being actually vanished, in which case they can have no proportion, nor yet as being of any real magnitude, when their proportion cannot be the same with the proportion of the fluxions, but by supposing that there can be represented to the mind some intermediate state of these augments or decrements at the very instant in which they vanish."

Robins discusses at considerable length the contention that the varying quantity must reach its limit, maintaining that it rests on a misconception of the phrase "given difference," and that even if absolute coincidence of a varying magnitude and its limit could occur, that circumstance is quite irrelevant.

Philalethes took up his defence in the number for January 1736, in an article entitled *Considerations occasioned by a Paper in the last Republic of Letters*. He begins by acknowledging in handsome terms that Robins has established beyond all doubt or cavil the truth of Newton's rule for finding fluxions, but he maintains that Robins misrepresents Newton, his own exposition being in accordance with the words of their common master. He accepts the description quoted above from the *Review*, provided the *intermediate state* of the augments is not interpreted to mean the *quantity* or *magnitude* of the augments. He thinks that *state* "may be represented to the mind, and

conceived by contemplating this proportion, not in the vanishing quantities themselves, but in other quantities permanent and stable, which are always proportional to them." Philalethes is here, I think, at his best; but he almost immediately, and one might say necessarily, involves himself in the quite gratuitous difficulty of undertaking to prove that Lemma I, Sect. 1. Book I of the *Principia*, necessarily requires that the variable must reach its limit. His contention rests on the literal interpretation of the words *fiunt ultimo aequales*, and on the idea that the words *data quavis differentia* do not mean *a difference first assigned*, according to which the degree of approach of the varying quantity may be afterwards regulated. He repeats from the November paper an illustration designed to show that the inscribed figure Lemma 2 does ultimately coincide with the curvilinear figure, but, as may be supposed, it is impossible to represent to the mind the *last* form of the inscribed figure which is to be equal to the curve. In any case, the whole conception is identical with that on which the method of indivisibles is founded and quite distinct from that of limits. Had Philalethes been able to see that the word *equal*, when associated with the restrictive adverb *ultimately*, was used in a wider meaning than in elementary mathematics, he might have been spared the trouble of all the controversy, and would have left to posterity a better estimate of his capacity than he has actually done.

The next contribution by Robins is a paper in the number of the *Republic* for April, 1736, entitled *A Dissertation showing that the Account of the Doctrines, &c., is agreeable to the real Sense and Meaning of their great Inventor.* The *Dissertation* is a somewhat lengthy document; it goes over the whole field, comparing in clear and interesting form the methods of exhaustions, indivisibles, and prime and ultimate ratios; points out Newton's early use of indivisibles, but shows how his discovery of the method of prime and ultimate ratios furnished a solid foundation for his demonstrations; and proves, as I think, beyond all possibility of doubt that his own account of Newton's aim and methods is thoroughly accurate. He fairly faces the objections Berkeley had raised, and shows with great skill that these are due to misconceptions of Newton's terminology, which disappear when that terminology is examined in the light of Newton's demonstrations and explicit cautions. He fully acknowledges the imperfection of certain phrases, taken by themselves, as, for example, when he says towards the close, "the ultimate ratio of variable quantities, the ratio with which quantities vanish, are, in strict propriety

of speech, figurative expressions: nay, the last form of a figure, and the form wherewith a figure vanishes, might be interpreted upon the foot of indivisibles. But here these phrases only signify the limits, &c." In the course of the work he departs from his previous practice by explicitly naming Philalethes, and quoting from that gentleman's writings passages to which he objects.

It is perhaps worth noting that he frequently quotes from the account of the *Commercium Epistolicum* in the *Philosophical Transactions* as a document written by Newton himself. Philalethes subsequently asks the authority for attributing that account to Newton, and though Robins gives no authority, both he and Wilson, the editor of his *Tracts*, continue to cite it as Newton's own work.

The controversy between these two disputants might well have closed with the *Dissertation*, but the pugnacity of Philalethes, which was worthy of a better cause, could not brook such a termination. He returns to the attack in two long articles contributed to the numbers of the *Republic* for July and August, 1736, with the title, *Considerations upon some passages of a Dissertation, &c.* These articles display considerable ingenuity of the kind usually associated with a pettifogging lawyer, and great facility in the composition of Latin verse; but they equally reveal his ignorance of mathematical history, and the poverty of his mathematical attainments, while they are disfigured by personalities that verge at times on indecency. They really contribute nothing to the elucidation of the points in dispute. Robins replied in the numbers for August and September (Appendix) in articles with the title *Remarks on the Considerations, &c.*, keeping clear of personalities, and confining himself to answering the *Considerations* in short notes to its chief paragraphs.

To the *Remarks* of Robins there is a long rejoinder by Philalethes in an Appendix to the *Republic* for November, but on this rejoinder Robins offers no criticism. He inserts, however, an *Advertisement* (a word which must not be taken in its modern meaning) in the December number, in which he says: "I think it a very ill compliment from me to the publick to undertake a serious answer to the unadvised speeches of an angry man." He "takes leave of Philalethes with this observation only upon the two points whereon our controversy does indeed solely depend; his definition of nascent and evanescent quantities, and his interpretation of Sir Isaac Newton's first Lemma relating to the doctrine of prime and ultimate ratios."

On the first point, he says Philalethes has relinquished those passages where the terms are expressly discoursed of, while in relation to the Lemma, he maintains that his opponent tries to defend his interpretation by taking refuge in the language of the first edition, altered by Newton himself in the later editions. He leaves him to explain himself on this Lemma to Dr Pemberton.

To understand this reference, it is necessary to explain that the Appendix to the September *Republic*, which contained Robins's *Remarks*, had a *Postscript* from Pemberton, occasioned by a passage in the August article of Philalethes, in which the latter had stated that Newton "by some means or other" had been "prevailed upon to change the word *perplexas* into *longas*" (*Principia*, Book I., Sect. 1. Scholium at end, *praemisi haec lemmata*, &c.) Pemberton resented the remark as an insinuation against himself as editor of the third edition, in which the change was made, and after defending his action as fully sanctioned by Newton, he took occasion to state that "he had the very best opportunity of knowing Sir Isaac's true mind," and he was "fully satisfied that Mr Robins (had) expressed Sir Isaac Newton's real meaning" in regard to the Lemma.

This Postscript was the signal for the renewal of the controversy, but now Pemberton took the place of Robins. It would be unprofitable to follow it further, so far as Philalethes is concerned, and it may be sufficient to state that the contributions of the two disputants will be found in an Appendix to the *Republic* for December 1736, and in the numbers from February to October 1737 of the "*History of the Works of the Learned*"—a monthly magazine which was formed from the amalgamation of the *Republic* and another journal *The Literary World*.

An interesting article by Pemberton in the number for January 1741 of the *Works of the Learned* may be referred to. In that article he interprets the famous Lemma in the sense of Robins, and quotes a passage from Gregory of St Vincent (*Def.* 3 *Libri de Progressionibus Geometricis*) as a probable origin for Newton's terminology of *ultimate ratio*—that is, it was borrowed from the language used in geometrical progressions. The language of Newton's Lemma is strikingly similar to that of Gregory.

Besides the controversy on Fluxions above described, there was another between Robins and Jurin, occasioned by a review which the former had published of an essay by Jurin *Upon*

Distinct and Indistinct Vision, appended to Smith's *Compleat System of Opticks* (1738). The only occasion for referring to this second controversy is that in the preface to a rejoinder published by Robins in 1740, Robins gives a full statement of the reasons which induced him to confine his first publications on Fluxions to an impersonal statement of its principles. Suspecting from the first that Philalethes was Dr Jurin, he did not wish to appear as an opponent of one whom he considered as a personal friend, and he had even submitted his earlier manuscripts to Jurin, through a common acquaintance, with the object of suppressing anything that might seem to reflect on Philalethes. The extravagance, however, of the later articles of Philalethes had obliged Robins to refer to him explicitly, and in this preface it is established beyond all reasonable doubt that Jurin and Philalethes were the same person.

It is impossible to pass from this stage of the controversy evoked by the *Analyst*, without referring to the contemporary estimate of the merits of Jurin and Robins. It is usually Jurin who obtains the credit of refuting Berkeley, and when Robins is mentioned at all, his criticism is put alongside that of Jurin. Thus, Maclaurin, in the Preface to his *Fluxions*, names Philalethes and Robins, as two who had undertaken the defence of the method of Fluxions, but nowhere so far as I know, has he tried to reconcile their divergent interpretations. John Stewart, Professor of Mathematics in Aberdeen, published in 1745, a translation of the *Quadratura*, and the *Analysis per aequationes*, accompanying the translation with a voluminous and in many respects extremely able commentary; yet, he quotes Philalethes as having demolished the arguments of the author of the *Analyst*, but never, so far as I can discover after a diligent study of his work, even names Robins. Again, Wilson, in an Appendix to his edition of Robins's *Tracts* (1761) refers to a French translation of Newton's *Fluxions* by M. de Buffon, in which the translator in his Preface abuses Robins in violent terms, and represents Philalethes as having completely triumphed over him.

It seems to me beyond doubt, that if Philalethes has correctly interpreted Newton, the latter has no claim to be considered as the one who first established the Calculus on the basis of Limits, and that it is Robins who should get the credit of so founding the Calculus. From first to last, Philalethes uses the language of Newton in the sense of the writers on indivisibles, and is totally unable to comprehend the utter inconsistency that he thus introduces into Newton's writings.

The protests that Newton made in the *Principia*, in the *Quadratura*, and throughout the Priority Controversy, are absolutely meaningless in the view Philalethes takes of Newton's doctrine. There can be no question, that there is a profound difference of conception in the views of Philalethes and Robins, and I confess myself at a loss quite to understand the favour shown to the work of Philalethes, and the comparative neglect of the brilliant essays of Robins. It is impossible by means of extracts to convey a sufficient sense of the extreme vagueness and want of precision on the part of Philalethes when treating the crucial points of a theory of limits; his language is largely figurative, and can have definite meaning assigned to it only by a re-interpretation based on a clear conception of a limit. Robins, on the other hand, is a ripe student of the masterpieces of antiquity, and is as precise and definite as Archimedes himself. His admiration for Newton is great, but it is sane, and the defence he makes of Newton's work is perhaps the best in existence.

As has been already stated, the arguments of the *Analyst* were, in a more or less explicit form, discussed by most English writers on Fluxions for several years after the publication of that work. There is, however, only one treatise that may here be noted as a direct fruit of the strictures of Berkeley, namely the great treatise by Maclaurin. This work was published at Edinburgh in 1742, but the greater part of the First Book was printed in 1737. Maclaurin's *Fluxions* is too well known to justify me in doing more than refer to it. The introductory chapters of Book I. are, I think, rendered rather tedious through his desire to stick closely to the ancient geometric method, but there can be no question of the immense power of logical exposition they display. It may be noted, as an instance of his breadth of mind, that he has even a good word to say for the Leibnizian calculus, and he shows how the demonstrations by that method may be made thoroughly rigorous.

Berkeley did a great service to sound reasoning in mathematics by the publication of the *Analyst*. The rapid accumulation of results, due to the introduction of the new analysis, had tended to throw into the background the logical principles on which any truly scientific knowledge of mathematics can alone be based, and the controversy the *Analyst* called forth is favourably distinguished from that on the invention of the Calculus by the comparative absence of the grosser personalities. Were it for nothing else than the *Discourse* and *Dissertation*

of Robins, and the *Fluxions* of Maclaurin, Berkeley's name should be had in reverence of mathematicians.

One may almost regret that no equally gifted critic of the theory of Infinite Series stood out to challenge the work of mathematicians in that department of their science. Mr Cantor's volume records the multitudinous results that were rapidly finding their way into publicity, * but it shows at the same time how badly needed was a logical analysis of the nature of an infinite series. But, after all, it was perhaps better in the end that this formal period ran its course, for when, at a later time, the necessary revision came, there was no lack of instances with which to illustrate the necessity for the restrictions to which the employment of such series is subject.

As it is to the Edinburgh Mathematical Society this paper is being read, I may be allowed to make a personal reference. My interest in the *Analyst* controversy was originally awakened by the lamented and gifted A. Y. Fraser, the first Secretary of the Society. His interest in it had been aroused, when he was an undergraduate at Aberdeen, by current unsatisfactory demonstrations of the leading propositions in the calculus, and he always attributed to Robins his first acquaintance with the true meaning of the theory of limits. How thoroughly he had profited by his study of Robins may be seen by the article on the Calculus, which he contributed to Chambers's *Encyclopaedia*. It was, I know, his intention to contribute to our *Proceedings* an account of the *Analyst* controversy, and at the time of his death he was engaged in arranging the materials; so far as I have been able to ascertain, however, there has not been found among his papers any record of his work in this field. How great a loss we have thus sustained, those who knew Mr Fraser's brilliant powers of exposition can best appreciate.

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* At p. 654 Mr Cantor expresses the opinion that in a certain memoir of Euler the idea on which the method of undetermined coefficients is founded is first clearly expressed. The method of undetermined coefficients was certainly well known to Newton, and is frequently applied in the *Quadratura*.

Third Meeting, January 13th, 1899.

ALEXANDER MORGAN, Esq., M.A., D.Sc., President, in the Chair.

Note on the Inequality Theorems that lead up to the Exponential Limit

$$\lim_{x=\infty} \left(1+\frac{1}{x}\right)^x = e.$$

By CHARLES TWEEDIE, M.A., B.Sc., F.R.S.E.

The object of the paper is to suggest a rearrangement of these theorems, which, it is contended, is simpler than, but as effective as, that usually adopted. The slightly more general theorems thus obtained suggest themselves readily, but the writer has not endeavoured to investigate other than the most elementary theorems in question.

THEOREM I. If $a_1, a_2, \ldots a_n$ be n positive quantities then

$$\frac{\Sigma a}{n} \not< \sqrt[n]{\Pi a}.$$

The theorem is true for any two quantities a_1, a_2, or for three a_1, a_2, a_3. Assume it true for any $n-1$ quantities $a_1, a_2, \ldots a_{n-1}$.

Arrange $a_1 \ldots a_n$ in order of magnitude, a_n being greatest, and let $\sqrt[n-1]{a_1 \ldots a_{n-1}} = b$, so that $b < a_n$.

$$\therefore \quad \frac{a_1 + \ldots + a_{n-1}}{n-1} \not< b,$$

$$\therefore \quad a_1 + \ldots a_{n-1} \not< (n-1)b.$$

$$\text{Hence} \quad a_1 + \ldots + a_n \not< a_n + (n-1)b,$$

$$\therefore \quad \frac{a_1 + \ldots + a_n}{n} \not< b + \frac{(a_n - b)}{n};$$

$$\therefore \quad \left(\frac{a_1 + \ldots + a_n}{n}\right)^n \not< \left(b + \frac{a_n - b}{n}\right)^n$$

$$\not< b^n + {}_nC_1 b^{n-1} \frac{a_n - b}{n} + \ldots$$

$$\not< a_n b^{n-1} + \ldots$$

$$\not< \Pi a + \ldots$$

$$\therefore \quad \frac{\Sigma a}{n} \not< \sqrt[n]{\Pi a} \quad \therefore \text{ etc.}$$

(*Vide* Weber's Algebra.)

THEOREM II. If a, m, and n be positive then

$$\frac{ma^x + na^y}{m+n} > a^{\frac{mx-ny}{m-n}} \quad \text{(i.)}$$

and $$\frac{ma^x - na^y}{m-n} < a^{\frac{mx-ny}{m-n}} \quad \text{(ii.)}.$$

(i.) is a direct consequence of I. as found by putting m a's equal to a^x, n of them equal to a^y. This proves the theorem when m and n are positive integers; but from the manner in which m and n occur it is clear that we can substitute for them any fractions so that the theorem is true when m and n are not integers.

From (i.) we deduce, when $m-n$ is positive,

$$\frac{(m-n)a^{\frac{mx-ny}{m-n}} + na^y}{(m-n)+n} > a^{\frac{(mx-ny)+ny}{(m-n)+n}}$$

$$> a^x;$$

$$\therefore \quad (m-n)a^{\frac{mx-ny}{m-n}} > ma^x - na^y,$$

$$\therefore \quad \frac{ma^x - na^y}{m-n} < a^{\frac{mx-ny}{m-n}}.$$

Since x and y are *any* quantities it is clearly immaterial whether $m-n$ or $n-m$ be positive.

(This theorem was suggested by Professor Chrystal, though not in this form.)

THEOREM III. If a be positive and distinct from unity, and $m-n$ positive, then

$$\frac{a^m - 1}{m} > \frac{a^n - 1}{n}.$$

This theorem is of considerable importance in the borderland between the elementary analysis and the infinitesimal calculus.

(a) Let m and n be both positive.

In (ii.) of the preceding put $x = n$, $y = m$, so that $mx - ny = 0$. Then (ii.) becomes

$$\frac{ma^n - na^m}{m-n} < 1,$$

$$\therefore \quad ma^n - na^m < m-n, \quad \because\ m-n \text{ is positive.}$$

$$\therefore \quad m(a^n - 1) < n(a^m - 1),$$

$$\therefore \quad \frac{a^m - 1}{m} > \frac{a^n - 1}{n},$$

which proves the theorem when m and n are positive.

(β) Since a is any positive quantity, we deduce

$$\frac{\left(\frac{1}{a}\right)^m - 1}{m} > \frac{\left(\frac{1}{a}\right)^n - 1}{n},$$

$$\therefore \frac{a^{-m} - 1}{-m} < \frac{a^{-n} - 1}{-n},$$

$$i.e. \quad \frac{a^{\nu} - 1}{\nu} < \frac{a^{\mu} - 1}{\mu},$$

if μ and ν be both negative and $\mu - \nu = (-n) - (-m) = m - n$ be positive.

(γ) Finally if we put $x = -n$, $y = m$ in II., we obtain

$$\frac{ma^{-n} + na^m}{m+n} > 1,$$

$$\therefore \quad ma^{-n} + na^m > m + n,$$

$$\therefore \quad \frac{a^m - 1}{m} > \frac{1 - a^{-n}}{n}, \quad i.e. \quad > \frac{a^{-n} - 1}{-n}.$$

The theorem is therefore true in all cases.

Cor. 1. If a be any positive quantity $\dfrac{a^x - 1}{x}$ constantly increases with x.

Cor. 2. $\dfrac{a^x - 1}{x} \lessgtr a - 1$ according as $x \lessgtr 1$.

THEOREM IV.

$$ma^{m-1}(a-1) > a^m - 1 > m(a-1)$$

for all values of m except those for which $0 < m < 1$, when the signs of inequality are reversed.

By theorem III. we have in all cases

$$\frac{a^m - 1}{m} > \frac{a^{m-1} - 1}{m-1}$$

Hence if $m(m-1)$ be positive we deduce

$$(m-1)(a^m - 1) > m(a^{m-1} - 1),$$

$$i.e. \quad ma^{m-1}(a-1) > a^m - 1.$$

But if m be positive and less than unity, $m(m-1)$ is negative and

$$(m-1)a^{m-1} < m(a^{m-1} - 1),$$

so that

$$ma^{m-1}(a-1) < a^m - 1.$$

To prove the second part of the inequality we note that when m is positive

$$\frac{a^m-1}{m} \lesseqgtr \frac{a-1}{1} \quad \text{according as } m \lesseqgtr 1;$$

and if m be negative

$$\frac{a^m-1}{m} < \frac{a-1}{1}$$

$$\therefore \quad a^m-1 > m(a-1) \quad \because \ m \text{ is negative.}$$

Hence in all cases

$$ma^{m-1}(a-1) \gtreqless a^m-1 \gtreqless m(a-1)$$

according as m does not, or does, lie between 0 and 1.

THEOREM V. $\underset{x=\infty}{\mathrm{L}} \left(1+\frac{1}{x}\right)^x$ is finite.

If x and y be positive; $\frac{x}{y}$ positive and greater than 1; and $1\pm\frac{1}{x}$, $1\pm\frac{1}{y}$ be positive, we deduce

$$\frac{\left(1\pm\frac{1}{x}\right)^{\frac{x}{y}}-1}{\frac{x}{y}} > \frac{\left(1\pm\frac{1}{x}\right)-1}{1},$$

$$\therefore \quad \left(1\pm\frac{1}{x}\right)^{\frac{x}{y}}-1 > \pm\frac{1}{y},$$

$$\therefore \quad \left(1\pm\frac{1}{x}\right)^{\frac{x}{y}} > 1\pm\frac{1}{y}.$$

Hence $$\left(1\pm\frac{1}{x}\right)^x > \left(1\pm\frac{1}{y}\right)^y$$

and $$\left(1\pm\frac{1}{x}\right)^{-x} < \left(1\pm\frac{1}{y}\right)^{-x}.$$

Hence, $$\left(1+\frac{1}{x}\right)^x > \left(1+\frac{1}{y}\right)^y$$

and $$\left(1-\frac{1}{x}\right)^{-x} < \left(1-\frac{1}{y}\right)^{-y}, \text{ if } x > y;$$

and we have the conclusion that

$\left(1+\frac{1}{x}\right)^x$ constantly increases with x; and $\left(1-\frac{1}{x}\right)^{-x}$ constantly decreases as x increases.

On the other hand, when y is > 1,

$$\frac{\left(1+\frac{1}{x}\right)^{-\frac{x}{y}}-1}{-\frac{x}{y}} < \left(1+\frac{1}{x}\right)-1$$

$$\therefore \left(1+\frac{1}{x}\right)^{-\frac{x}{y}}-1 > -\frac{1}{y}$$

$$\left(1+\frac{1}{x}\right)^{-x} > \left(1-\frac{1}{y}\right)^{y}$$

$$\therefore \left(1+\frac{1}{x}\right)^{x} < \left(1-\frac{1}{y}\right)^{-y}.$$

Hence $\left(1+\frac{1}{x}\right)^{x}$ has some definite upper limit A,

and $\left(1-\frac{1}{x}\right)^{-x}$ a definite lower limit B, where $A \not> B$

As a matter of fact, $A = B$ for

$$\underset{x=\infty}{\mathrm{L}} \frac{\left(1-\frac{1}{x}\right)^{-x}}{\left(1+\frac{1}{x}\right)^{x}} = \mathrm{L}\left(1-\frac{1}{x^2}\right)^{-x} = \mathrm{L}\left[\left(1-\frac{1}{x^2}\right)^{-x^2}\right]^{\frac{1}{x}}$$

$$= B^0 = 1.$$

Even if this paper had no other interest, it would draw the attention to the remarkable fact that the *order* in which these theorems of inequalities require to be demonstrated is somewhat arbitrary.

Thus Theorem V. may be utilised to prove Theorem I. (*vide* Chrystal's *Algebra*), and yet the inequality

$$\left(1+\frac{1}{x}\right)^{x} > \left(1+\frac{1}{y}\right)^{y}, \quad x > y,$$

must be very perfect, for, from the inequality

$$\left(\frac{a+c}{a-c}\right)^{a} < \left(\frac{b+c}{b-c}\right)^{b}, \quad a > b > c > 0,$$

may readily be concluded that

$$\left(1+\frac{1}{x}\right)^{x+\frac{1}{2}} < \left(1+\frac{1}{y}\right)^{y+\frac{1}{2}}.$$

It may be added that the process of differentiation may be successfully applied to III., and to the inequalities of V.

Against Euler's Proof of the Binomial Theorem for Negative and Fractional Exponents.

By R. F. Muirhead, M.A., B.Sc.

In the proof of the binomial theorem for negative and fractional indices given in many text-books of algebra, and attributed to Euler, one step seems to me to involve a very gross assumption.

The symbol $f(m)$ having been taken to denote the series

$$1+mx+\frac{m(m-1)}{1.2}x^2+\frac{m(m-1)(m-2)}{1.2.3}x^3+\ldots\ldots$$

it is pointed out that whenever m and n are positive integers we know that $f(m)\times f(n)\equiv f(m+n)$;

and the conclusion is drawn that since this is true for all positive integral values of m and n, by the "permanence of equivalent forms" (whatever that may mean) we can conclude that it is true also for negative and fractional values of m and n, whenever $f(m)$ and $f(n)$ are convergent.

If instead of the above definition of $f(m)$ we take $f(m)$ to denote

$$\left\{1+mx+\frac{m(m-1)}{1.2}x^2+\ldots\right\}\left\{1-\phi(x)\sin m\pi\right\}+\psi(x)\sin m\pi,$$

$\phi(x)$ and $\psi(x)$ being arbitrary functions, we have, when m is an integer, $f(m)=(1+x)^m$ and by exactly the same reasoning as that referred to we can conclude that in this case also $f(m)\times f(n)\equiv f(m+n)$ for all values of m and n, which is obviously false for most values of $\phi(x)$ and $\psi(x)$.

There must be a flaw in the reasoning: and that clearly is the assumption that an identity true for an infinite number of values of the arguments under certain restrictions as to those values, is necessarily true when these restrictions are removed. Simple examples showing the falsity of the assumption could easily be multiplied: thus $\sin(\theta+m\pi)=\cos m\pi\sin\theta$ for *all* integral values of m. But the equation is not true for fractional values of m.

The retention of the bad reasoning in some of our text-books is the less to be excused, as there exists an unexceptionable proof by means of the theorem of Vandermonde, as given in C. Smith's *Algebra* and others.

If circumstances should render it impracticable in any case to assume a knowledge of Vandermonde's Theorem in teaching the Binomial Theorem, then one might adopt the honest course of confessing the gap in the proof, making no attempt to juggle with the "principle" of the "permanence of equivalent forms," but merely pointing out that the identity $f(m) \times f(n) \equiv f(m+n)$ can be verified to as many terms as we please.

Note on Continued Fractions.

By R. F. Muirhead, M.A., B.Sc.

1. It is usual to deduce most of the properties of Continued Fractions from the recurrence formulas

$$p_n = a_n p_{n-1} + p_{n-2}$$

$$q_n = a_n q_{n-1} + q_{n-2},$$

connecting successive convergents of the fraction

$$a_1 + \frac{1}{a_2 +} \quad \frac{1}{a_3 +} \cdots\cdots .$$

In this Note, I treat one or two theorems from a different point of view.

It is very obvious, by the formation of the Continued Fraction, that the convergents are alternately greater and smaller than those preceding and following them; the formal proof may be put as follows: —

In the Continued Fraction

$$Y = a_1 + \frac{1}{a_2 +} \quad \frac{1}{a_3 +} \cdots\cdots \frac{1}{a_n + y},$$

where y for the present is arbitrary, writing

$$Y_{n-1} \equiv a_n + y, \qquad Y_{n-2} \equiv a_{n-1} + \frac{1}{Y_{n-1}}, \qquad Y_{n-3} \equiv a_{n-2} + \frac{1}{Y_{n-2}} \cdots\cdots\cdots\cdots$$

$$Y_1 \equiv a_2 + \frac{1}{Y_2}, \qquad Y \equiv a_1 + \frac{1}{Y_1};$$

it is obvious that as y increases Y_{n-1} increases,

as Y_{n-1} increases Y_{n-2} decreases,

as Y_{n-2} decreases Y_{n-3} increases,

and so on, so that Y increases with y if n is odd, but if n is even Y decreases as y increases.

Now consider the fraction

$$X \equiv a_1 + \frac{1}{a_2 +} \quad \frac{1}{a_3 +} \cdots\cdots$$

and let $\qquad X \equiv a_1 + \frac{1}{X_1}, \qquad X_1 = a_2 + \frac{1}{X_2}, \qquad$ etc.

(as in Chrystal's *Algebra*), and let p_1/q_1, $p_2/q_2 \ldots$ be the convergents of the fraction X. We see that corresponding to the values

$$y = 0, \qquad y = \frac{1}{X_n}, \qquad y = \frac{1}{a_{n+1}},$$

we have the values $\quad Y = p_n/q_n, \quad Y = X, \quad Y = p_{n+1}/q_{n+1}.$

Hence $\frac{p_n}{q_n}$, X, and $\frac{p_{n+1}}{q_{n+1}}$ *are in ascending or descending order of magnitude according as* n *is odd or even.*

2. To show by this method that each convergent is closer to X than the preceding one, we need this Lemma :

If a, b, c are three positive magnitudes such that b lies between c and a, but nearer to c, and also $1/b$ lies nearer to $1/c$ than to $1/a$, then the three magnitudes

$$A \equiv m + \frac{r}{a}, \qquad B \equiv m + \frac{r}{b}, \qquad C \equiv m + \frac{r}{c}$$

satisfy the same conditions among themselves, m and r being any positive quantities.

For obviously B lies between A and C.

Then $$\frac{B\sim C}{A\sim B}=\left(\frac{1}{c}\sim\frac{1}{b}\right)\div\left(\frac{1}{b}\sim\frac{1}{a}\right).$$

which by hypothesis is < 1, so that B is nearer to C than to A.

Again $$\left(\frac{1}{C}\sim\frac{1}{B}\right)\div\left(\frac{1}{B}\sim\frac{1}{A}\right)=\frac{A(B\sim C)}{C(A\sim B)}$$

$$=\frac{m+\dfrac{r}{a}}{m+\dfrac{r}{c}}\cdot\frac{\dfrac{1}{c}\sim\dfrac{1}{b}}{\dfrac{1}{b}\sim\dfrac{1}{a}}=\frac{am+r}{cm+r}\cdot\frac{b\sim c}{a\sim b}.$$

The last formula shows that the ratio is <1 when $a\gtrless c$.

The last but one shows that the ratio is <1 when $a\lessgtr c$.

Thus the Lemma is completely proved.

Now let Y', Y'', Y''' denote the values of Y when y has the values 0, $\dfrac{1}{X_n}$ and $\dfrac{1}{a_{n+1}}$, and similarly for Y_1, Y_2, etc.

Thus $Y'_{n-1}=a_n$, $Y''_{n-1}=a_n+1/X_n$, $Y'''_{n-1}=a_n+1/a_{n+1}$.

It is easy to prove that Y'_{n-1}, Y''_{n-1}, Y'''_{n-1} satisfy the same conditions as the quantities a, b, c of the Lemma. Hence, as proved in the Lemma, Y'_{n-2}, Y''_{n-2}, Y'''_{n-2} also satisfy these conditions, and so on for the other Y's. Finally we find that Y', Y'', Y''', which are respectively p_n/q_n, X, and p_{n+1}/q_{n+1}, satisfy these conditions.

3. Thus we have proved directly, without the aid of the recurrence formulae, two of the most important properties of continued fractions. The investigation in the previous paragraphs is obviously applicable to the General Continued Fraction $a_1+\dfrac{b_2}{a_2+}\ \dfrac{b_3}{a_3+}\cdots$ so long as the a's and b's are positive.

The form of the investigation has implied that the continued fraction is terminating; but we can see that it will hold good for an infinite continued fraction, provided it converges to a definite limit.

A Proof of the Binomial Theorem when the Exponent is a Positive Integer.

By R. F. Muirhead, M.A., B.Sc.

$$\frac{(1+x)^n-1}{(1+x)-1}=(1+x)^{n-1}+(1+x)^{n-2}+(1+x)^{n-3}+\ldots+(1+x)+1;$$

$$\therefore\ (1+x)^n-1=x\{(1+x)^{n-1}+(1+x)^{n-2}+\ldots+(1+x)+1\} \quad \text{(i)}$$

Thus $(1+x)^n-1$ is divisible by x.

Hence $(1+x)^n=1+$ terms in x and its powers.

Then (i) may be written:

$$(1+x)^n=1+x\{n+\text{ terms in } x \text{ and its powers}\}$$

$$=1+nx+\text{ higher powers of } x. \quad \text{(ii)}$$

Now $(a+b)^n=a^n+(n,\ 1)a^{n-1}b+(n,\ 2)a^{n-2}b^2+\ldots+b^n,$

where $(n,\ 1)$, $(n,\ 2)\ldots$ are numerical coefficients independent of a and b, and by (ii) we see that $(n,\ 1)=n$.

Again $(1+x+y)^n=(1+x)^n+n(1+x)^{n-1}y+(n,\ 2)(1+x)^{n-2}y^2+\ldots$ (iii)

Also $(1+x+y)^n=x^n+nx^{n-1}(1+y)+\ldots+(n,\ r)x^{n-r}(1+y)^r+\ldots$ (iv)

The coefficient of $x^{n-r}y$ in (iii) is $n.(n-1,\ r-1)$;

the coefficient of $x^{n-r}y$ in (iv) is $(n,\ r).r$.

$$\text{Hence}\quad (n,\ r)=\frac{n}{r}(n-1,\ r-1)$$

$$=\frac{n}{r}\cdot\frac{n-1}{r-1}(n-2,\ r-2)$$

$$=\frac{n}{r}\cdot\frac{n-1}{r-1}\cdots\cdots\frac{(n-r+1)}{1}.$$

$$\text{Thus}\quad (a+b)^n=a^n+na^{n-1}b+\frac{n.\overline{n-1}}{1.2}a^{n-2}b^2+\ldots$$

$$+\frac{n.\overline{n-1}\ldots\overline{n-r+1}}{1.2\ \ldots\ r}a^{n-r}b^r+\ldots+b^n.$$

Fourth Meeting, 10th February 1899.

ALEXANDER MORGAN, Esq., M.A., D.Sc., President, in the Chair.

On the Eight Queens Problem.

By T. B. SPRAGUE, M.A., LL.D., F.R.S.E.

This is the problem discussed in my paper bearing the not very happy title "On the different non-linear arrangements of eight men on a chess-board", which was read to the Edinburgh Mathematical Society on 14th March 1890, and is printed in its *Transactions*, Vol. VIII, p. 30. At that time I was not aware that the problem had been discussed by any previous writer, and I treated it as an entirely new one. I have since learnt that a good deal has been written about it, and I propose on the present occasion to give briefly the history of the problem, and the results which have been arrived at; also to communicate some new results which I have obtained.

For those who are familiar with the game of chess, the best statement of the problem is as follows:—To find the different ways in which eight queens may be arranged on a chess-board, so that no two of them shall be in check of each other. For those who know nothing of that game, the problem may be stated thus:—To find the different ways in which eight pieces may be placed on a chess-board, so that no line joining two of them, shall be parallel either to a side or to a diagonal of the board.

My knowledge of the history of the problem is mostly derived from the following sources:—

1. Mr J. W. L. Glaisher's paper "On the Problem of the Eight Queens" in the *Philosophical Magazine* for December 1874. This is the work of an accomplisht mathematician, but the treatment of the problem is not entirely satisfactory, the explanation of the processes being often too brief: at all events, I found that much careful study was necessary before I could entirely understand it.

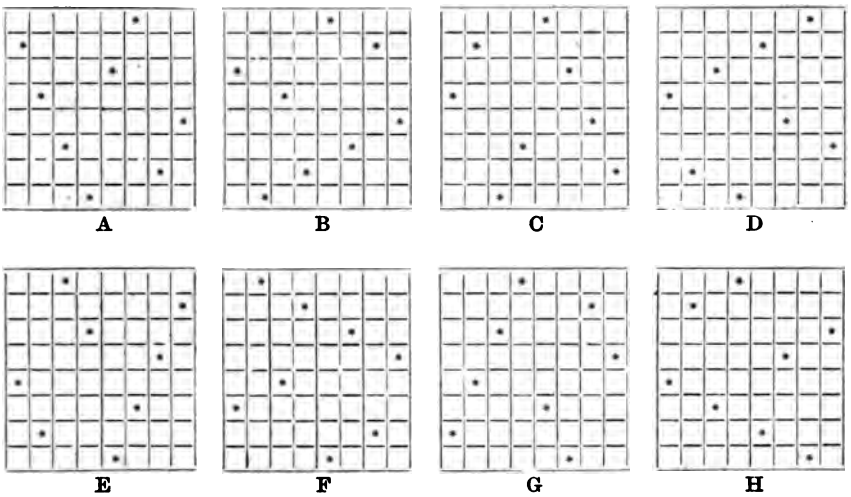
2. "Mathematical Recreations and Problems", by W. W. Rouse Ball (Macmillan & Co., 2nd Edition, 1892). This contains, on pages 85–88, a brief, and not entirely accurate, summary of the history of the problem, and of the results obtained.

3. Dr Pein's work entitled "Aufstellung von n Königinnen auf einem Schachbrett von n^2 Feldern, derart dass keine von einer andern geschlagen werden kann. (Von $n=4$ bis $n=10$). Von Dr August Pein, Oberlehrer." This bears the name of the printer, B. G. Teubner, Leipzig; but seems not to have been publisht in the usual way, as it is headed "Städtische Realschule zu Bochum. Beilage zu dem Jahresbericht über das Schuljahr, 1888-89." It is a very lengthy paper, is illustrated by numerous woodcuts, and deals with the problem in a very complete manner; but it seems to me to contain a good deal of unnecessary detail.

The problem seems to have been first proposed by Dr Nauck in the Leipzig *Illustrated Times* of 1st June 1850, and the complete solution was given by him in the number for the following 22nd of Sept. Meantime, the question had attracted the attention of Gauss, and it is discussed in his publisht correspondence with Schumacher.

Mr Glaisher states that the problem was proposed by Nauck to Gauss, and that the latter, after finding the number of solutions to be 76 and then 72, ultimately arrived at the correct number, 92; and Mr Ball follows him in saying that the earliest solution was given by Gauss; but these statements appear to be erroneous. Dr Pein has given in his paper several extracts, both from Gauss's correspondence with Schumacher, and from Nauck's original papers; and it appears from these that Nauck first stated (1st June 1850) that there were 60 solutions, but on 22nd September he gave the correct number, 92; also that Gauss said, in his letter of 1st September, that the number was not 60 but 76; and on 12th September that this must be reduced to 72. In his letter of 27th September, he mentions that Nauck had stated that the correct number was 92, but expresses a doubt whether that might not, like the 60 previously given, have to be corrected. It is clear, therefore, that Gauss had not himself at that time verified this number. It is true that he gives the twelve fundamental solutions, from which the whole 92 can be deduced; but Nauck had previously explained fully how from any one solution that is not symmetrical, seven others can be deduced. Glaisher's authority for his statements appears to be a paper by Dr Siegmund Günther, entitled "Zur Mathematischen Theorie des Schachbretts", which was contained in Grunert's "Archiv der Mathematik und Physik", and which, Glaisher says, gives an interesting account of the history of the problem. I have not seen this paper, and therefore cannot say whether Günther was responsible for the mistake.

As already mentioned, there are 92 ways in which the eight queens may be arranged so as to satisfy the required conditions; or 92 solutions of the problem. But these are not all independent of each other; for, if we have got any one solution, we can generally get three others by turning the chess-board through 1, 2, and 3 right angles. Four other solutions can then be got, by taking the reflection of the first solution in a mirror, and turning this through 1, 2, and 3 right angles. In this way eight solutions are got, which are so connected, that all eight can be obtained from any one of them by rotation and reflection.



The connection between the eight solutions, which may be called the different aspects of a single solution, will be best understood by considering the diagrams here given. Starting with the aspect A, we get B, C, D, by turning the board round in the opposite direction to the hands of a clock, through 1, 2, and 3 right angles respectively. Then E is got from A by reflection in a mirror passing through a vertical side of the square; and F, G, H are got by turning E similarly through 1, 2, and 3 right angles. We see also that, by the same kind of reflection, H may be got from B; G from C; and and F from D; so that the eight aspects may be arranged in the following pairs:—

A, E; B, H; C, G; D, F.

If the mirror passes through the top or bottom side of A, this gives us the aspect G; and by the same kind of reflection the aspects may be grouped in the following pairs:—

A, G; B, F; C, E; D, H.

If the mirror is parallel to the diagonal which runs from the left hand top corner (which we will call the first diagonal), reflection of A gives us F; and by similar reflections we can group the aspects in the following pairs:—

A, F; B, E; C, H; D, G,

Lastly, if the mirror is parallel to the other diagonal, reflection gives us the following pairs:—

A, H; B, G; C, F; D, E.

By reflecting the aspects alternately in two mirrors parallel respectively to a vertical side, and to the first diagonal, we get all the aspects from A, in the order

A, E, B, H, C, G, D, F.

By similar alternate reflections in mirrors parallel to a vertical side and the second diagonal, we get the eight aspects in the order

A, E, D, F, C, G, B, H;

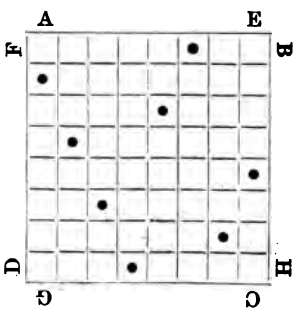
and by reflections in mirrors parallel to a horizontal side and to the first and second diagonals respectively, we get the eight aspects in the two orders

A, G, D, H, C, E, B, F,

and A, G, B, F, C, E, D, H.

The best way of representing any solution, is by writing down the figures that indicate the places of the queens in the various columns: thus, counting the squares from the top, solution A is represented by 24683175. The representations of the seven other aspects of this solution are most easily got by reading them off from the diagram.

A, 24683175
B, 38471625 = *rp*A
C, 42861357 = *ir*A
D, 47382516 = *ip*A
E, 57138642 = *i*A
F, 61528374 = *p*A
G, 75316824 = *r*A
H, 52617483 = *irp*A



It is not difficult to obtain from any one aspect of the solution the seven others, without a diagram or a chessboard: but in order to do this, it is convenient to arrange the aspects in a different order. From A we get, by inverting the order of the figures, E, which therefore I denote by iA. Next, substituting for each figure in A its difference from 9, we get G. This process, which corresponds to reflection in a mirror parallel to the horizontal sides, I call reversion; and I denote G by rA. Inversion of the figures gives us C, which I accordingly denote by irA. We have thus got the four aspects which are obtained by reflection in mirrors parallel to the sides of the board.

In order to find the other aspects, we must transform A in a more fundamental manner: we must, in fact, interchange the rows and columns. Thus in A

1 stands in the 6th place; therefore put 6 in the 1st place
2 „ „ „ 1st „ ; „ „ 1 „ „ 2nd „
3 „ „ „ 5th „ ; „ „ 5 „ „ 3rd „

and so on.

This process, which is the same thing as reflection in a mirror parallel to the first diagonal, I call perversion. When applied to A, it gives F, which I denote by pA. Inversion then gives us D, which is consequently ipA. Again, reversion of F gives B, which is therefore rpA; and lastly, inversion of B gives H, which is therefore irpA. The most suitable arrangement now is

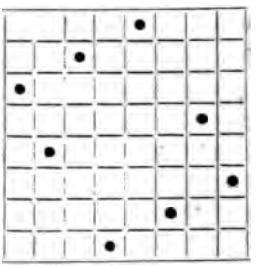
A,	24683175 = A	iA,	57138624 = E
rA,	75316824 = G	irA,	42861357 = C
pA,	61528374 = F	ipA,	47382516 = D
rpA,	38471625 = B	irpA,	52617483 = H

(It may be useful to give here a brief statement of the results we get by combining any two of these symbols of operation, i, r, and p. We have $i^2 = 1$, $r^2 = 1$, $p^2 = 1$; $ir = ri$; $ip = pr$; $rp = pi$.)

If it should happen that rotation through two right angles reproduces the original arrangement, then the solution will be symmetrical, and have only four aspects.

The 92 arrangements which satisfy the conditions of the problem, may thus be grouped under 12 distinct solutions, 11 of which have eight aspects each, while the twelfth is symmetrical and has only four. The 12 solutions (which were given by Gauss, but apparently not by Nauck) are represented by

15863724	25713864	26831475	*35281746
16837425	25741863	27368514	35841726
24683175	26178435	27581463	36258174



The symmetrical solution is 35281746, and is represented in the annext figure. The sum of each pair of figures equidistant from the beginning and end, 3, 6; 5, 4; 2, 7; 8, 1; is 9; and it is easily seen that this is the condition that an arrangement of eight men, representing a permutation of the numbers 1, 2,...8, may be symmetrical, or remain unaltered when turned round through two right angles. Such a solution I call "centric".

Gauss has given the following rule for testing whether the arrangement corresponding to any permutation, is a solution of the problem :—

To the successive figures in the permutation add the numbers 1, 2, 3,...8, respectively; then all the totals must be different. Again, add to the same figures the numbers 8,...3, 2, 1, respectively; then all these totals must also be different. Taking, for instance, the first solution, we find

1	5	8	6	3	7	2	4		1	5	8	6	3	7	2	4
1	2	3	4	5	6	7	8		8	7	6	5	4	3	2	1
2,	7,	11,	10,	8,	13,	9,	12		9,	12,	14,	11,	7,	10,	4,	5

He remarks that the problem may be stated without any reference to a chessboard, as follows:—Required to arrange the numbers 1, 2, 3,...8 in such an order that, if the successive figures are increast respectively by 1, 2, 3,...8, all the totals shall be different; and that, if they are increast respectively by 8,...3, 2, 1, these totals also shall all be different.

Gauss describes very clearly the process of systematic trial by which the problem can be solved. He begins with 1 in the first column, and 3 in the second, and tries to find a solution beginning with 13. By the conditions of the problem, neither 2 nor 4 is admissible in the third column, but only 5, 6, 7, or 8. We must therefore make trial of the beginnings

135.... 136.... 137.... 138....

Taking then 135, the conditions exclude 4 and 6 from the fourth place. Only 2, 7, 8, therefore, remain, and we have to make trial of the beginnings 1352..., 1357..., 1358....

Taking 1352, the conditions exclude 6 and 7 from the next place, so that the only beginnings that have to be tried are

13524... and 13528....

Having regard to the conditions, neither 6, nor 7, nor 8, can stand in the sixth place, after 13524, and this beginning therefore is to be rejected. Similarly the beginning 13528 is to be rejected, because neither 4, nor 6, nor 7, can come in the sixth place. The beginning 1352 is therefore inadmissible. Proceeding in the same way with 1357 and 1358, we find that both of these are also inadmissible. The beginning 135 is therefore inadmissible, and we have to make trial in a similar way of 136, 137, and 138.

It will be seen that this is exactly the same process as I described in my paper. The actual manipulation, however, adopted by Gauss differed from mine. He used cross-ruled paper, or a slate with lines cut pretty deeply in it; and he marked with pencil a cross (×) on each square as soon as it is supposed to be occupied by one of the Queens, and a nought (0) on each of the squares from which a Queen is thereby excluded; and he subsequently rubbed out, as the process proceeded, the marks that were not required. My plan is to place a pawn on each square supposed to be occupied by a Queen; and the removal of the pawns from the board is evidently a much easier process than rubbing out the marks.*

It is obvious that the problem is not restricted to the ordinary chess-board, containing eight squares in a side and sixty-four altogether, but applies to a board with n squares in a side, which therefore contains n^2 squares altogether; in which case, of course, n queens are to be arranged on it, so that no two of them shall be in check of each other.

In the year 1874 it was suggested by Dr Siegmund Günther that our problem might be solved by means of determinants (Grunert's *Archiv der Mathematik und Physik*, 1874, vol. 56, pp. 281–292; see also Günther's *Lehrbuch der Determinanten-Theorie für Studierende. Erlangen*, 1875. Ch. 2, § 11, p. 46).

* I found it was of great assistance in working with a board of 11 squares in a side, to have series of lines ruled on the board parallel to the two diagonals, and drawn with black and red ink alternately.

This suggestion was taken up and improved upon by Glaisher, in the paper mentioned above. He says:—

"Dr Günther remarks that if the determinant

$$\begin{vmatrix} a_1 & c_2 & e_3 & g_4 & k_5 & . & . & . & . \\ b_2 & a_3 & c_4 & e_5 & g_6 & . & . & . & . \\ d_3 & b_4 & a_5 & c_6 & e_7 & . & . & . & . \\ f_4 & d_5 & b_6 & a_7 & c_8 & . & . & . & . \\ h_5 & f_6 & d_7 & b_8 & a_9 & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & b_{2n-2} & a_{2n-1} \end{vmatrix}$$

"be expanded, and all the terms be rejected in which either the "same letter or the same suffix appears more than once, then the "terms that remain will give all the solutions of the problem. The "reason for the rule is evident: from the nature of a determinant "each term involves one constituent from each row and one from "each column, and the terms thus represent all the positions in "which the queens cannot take one another castle-fashion; the "omission of the terms in which the same letter or suffix appears "more than once, excludes the cases in which two or more queens "lie on the same diagonal (*i.e.*, can take one another bishop-fashion); "so that the terms that remain are the solutions. Dr Günther "develops the determinants for boards of 9, 16, and 25 squares, but, "owing to the number of terms involved, does not proceed further; "he remarks that for the chess-board of 64 squares it would be "necessary to calculate 20,160 terms."

Glaisher remarks that it would be quite out of the question to actually write down 20,000 terms; and he points out various ways of shortening the work, which render it unnecessary to write down all the terms of the determinant, as Günther apparently did. Taking, for instance, the board of 25 squares, for which the determinant becomes

$$\begin{vmatrix} a_1 & c_2 & e_3 & g_4 & k_5 \\ b_2 & a_3 & c_4 & e_5 & g_6 \\ d_3 & b_4 & a_5 & c_6 & e_7 \\ f_4 & d_5 & b_6 & a_7 & c_8 \\ h_5 & f_6 & d_7 & b_8 & a_9 \end{vmatrix}$$

Glaisher points out that, if a solution contains a_9, and we remove the row and the column containing a_9, we get a board of 16 squares,

on which the four queens can be arranged as required by the problem. If, then, we already know the solutions for $n=4$, we shall get from each of them which does not contain the letter a, a solution for $n=5$, by simply adding a_5. But it is easily seen that the only solutions when $n=4$ are

$$c_2e_3d_1b_4 \quad \text{and} \quad e_3b_2c_4d_1.$$

Hence we get two solutions for the 5-board,

$$c_2e_3d_1b_4a_5 \quad \text{and} \quad e_3b_2c_4d_1a_5,$$

which in the usual notation will be represented by 24135 and 31425. (It is to be borne in mind that Glaisher deals with the places of the queens in the several rows, not in the columns, as we have done above.)

These, however, are only aspects of the same solution, as each of them can be got from the other by reflection in the first diagonal. Glaisher then shows how to get the six solutions containing a_1, h_5, and k_5, respectively; but as, in consequence of the double symmetry of the solutions for the 4-board, all these are only other aspects of the solution we have already got, it is unnecessary to say anything more about them.

Whatever the value of n, a little consideration will show that, if we have all the solutions for the $(n-1)$-board, and in one of these solutions a diagonal of the board is open; that is to say, a queen does not stand on any square of the diagonal; then we shall get a solution for the n-board, by adding a new column and a new row, intersecting in that diagonal, and placing a queen on the new square thus added to it. We can do this for each end of the diagonal; so that, in general, we get two solutions for the n-board when a diagonal of the $(n-1)$-board is open; and if both diagonals are open, we get four solutions. These solutions Glaisher calls "ultimate" solutions; but I prefer to call them "corner" solutions.

Having thus got all the solutions that contain a corner square, Glaisher says that it would be useless to write down the terms of the determinant that contain one of the corner constituents, and we may therefore replace these by zeros, and take the simpler determinant:—

$$\begin{vmatrix} . & c_2 & e_3 & g_4 & . \\ b_2 & a_3 & c_4 & e_5 & g_6 \\ d_3 & b_4 & a_5 & c_6 & e_7 \\ f_4 & d_5 & b_6 & a_7 & c_8 \\ . & f_6 & d_7 & b_8 & \end{vmatrix}$$

In expanding the determinant the signs of the terms are immaterial, and we proceed as if all the terms were positive. We thus get

$$c_2 \begin{vmatrix} b_2 & c_4 & e_5 & g_6 \\ d_3 & a_5 & c_6 & e_7 \\ f_4 & b_6 & a_7 & c_8 \\ . & d_7 & b_8 & . \end{vmatrix} + e_3 \begin{vmatrix} b_2 & a_3 & e_5 & g_6 \\ d_3 & b_4 & c_6 & e_7 \\ f_4 & d_5 & a_7 & c_8 \\ . & f_6 & b_8 & . \end{vmatrix} + g_4 \begin{vmatrix} b_2 & a_3 & c_4 & g_6 \\ d_3 & b_4 & a_5 & e_7 \\ f_4 & d_5 & b_6 & c_8 \\ . & f_6 & d_7 & . \end{vmatrix}$$

Now if we have found all the solutions containing c_2, we can get those containing g_6, b_8, f_4, by turning the board through 1, 2, and 3 right angles, respectively; and those containing g_4, c_8, f_6, b_2, by reflection. Hence we may put each of these seven constituents equal to zero in the second and third terms. When this is done, all the constituents in the fourth row of the second term are replaced by zeros, and the determinant therefore vanishes. The third term, being multiplied by g_4 which is to be replaced by zero, also vanishes. Glaisher, however, does not notice that the second term vanishes, but proceeds:—In the first term, which is multiplied by c_2, we have to reject from the determinant every constituent which contains the letter c, or the suffix 2: or, in other words, we may put a zero in the place of every constituent that contains either c or 2. Similarly, in the second term, we reject every constituent which contains e or 3. We thus get the simplified determinants

$$c_2 \begin{vmatrix} . & . & e_5 & g_6 \\ d_3 & a_5 & . & e_7 \\ f_4 & b_6 & a_7 & . \\ . & d_7 & b_8 & . \end{vmatrix} + e_3 \begin{vmatrix} . & . & . & . \\ . & b_4 & c_6 & . \\ . & d_5 & a_7 & . \\ . & f_6 & b_8 & . \end{vmatrix}$$

(Here, it will be seen, Glaisher has omitted to put zeros for f_6 and b_8 in the second term; and this error is copied by Pein.)

The determinant which is multiplied by e_3, vanishes; and the other term becomes

$$c_2e_5 \begin{vmatrix} d_3 & . & . \\ f_4 & b_6 & . \\ . & d_7 & . \end{vmatrix} + c_2c_6 \begin{vmatrix} d_3 & a_5 & . \\ f_4 & . & a_7 \\ . & d_7 & b_8 \end{vmatrix}$$

putting zeros for a_5 and e_7 in the first of these minor determinants, and for b_6 in the second.

The first term vanishes, and the second gives us $c_2g_6a_5f_4b_8$, since we reject the combination $d_3a_7d_7$.

This solution, in the usual notation, is 25314, and is easily seen to be doubly symmetrical.

(The above explanation of the process is a little fuller than that given by Glaisher, which is so condenst as to be difficult to follow.)

If we employ in this case the method of systematic trial described by Guass, we shall arrive at the solutions more expeditiously. The corner solutions, if any, having been got from the solution for the 4-board, as explained above, we have to begin with 2; and we see at once that, by the conditions of the problem, this can only be followed by 4, or by 5. We now see that 24 can only be followed by 1, and this leads at once to 24135. This, however, is a corner solution; and is found to be only another aspect of a corner solution already found.

Next, beginning with 25, we have 251, or 253; and the former of these can only be followed by 4; but 2514 must be rejected, because 3 cannot come after it, according to the conditions. Passing on then to 253, we see at once that it can only be followed by 1, and we get the solution 25314.

It is useless to begin with 3, because any solution we should thus get, must have been already got, as it is obviously impossible that a solution should contain the middle points of all the 4 sides. It is also useless to begin with 4; since any solution beginning with 4 is another aspect of a solution, already got, beginning with 2.

Passing on now to the 8-board, or the ordinary chess-board, Glaisher gives no particulars as to his method of developing the determinant; but Pein, who has used Glaisher's method, tells us that, in order to get the 9 distinct solutions that begin with b_2, and d_3, he has had to investigate 26 minor determinants of the fifth degree. It will be noticed that he deals with the columns of the board, whereas Glaisher deals with the rows. Pein has solved the problem for the 9-board, and the 10-board; and he says that, in order to get the solutions that begin with b_2, d_3, and f_4, he has had to investigate, in the former case, 134 minor determinants of the fifth order; and in the latter, 296 minor determinants of the sixth order.

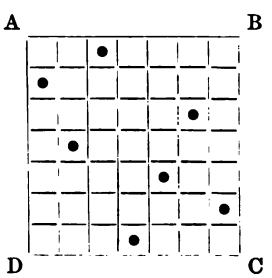
The labour of writing down and expanding so many determinants, will evidently be very great; and the use of determinants should possess some very decided countervailing advantages, to make it

preferable to the method of systematic trial, as explained by Gauss. After careful consideration, however, and trial of both methods, I think that the latter is greatly to be preferred. In order to compare the two methods, it will be convenient to show in some detail how the method of determinants is to be employed in the case of the ordinary chess-board.

I find it desirable to employ a more symmetrical form of determinant, than the one proposed by Günther and adopted by Glaisher; and I write the determinant for the ordinary chess-board as follows:—

$$\begin{vmatrix}
\triangle_1 & A_2 & B_3 & C_4 & D_5 & E_6 & F_7 & G_8 \\
a_2 & \triangle_3 & A_4 & B_5 & C_6 & D_7 & E_8 & F_9 \\
b_3 & a_4 & \triangle_5 & A_6 & B_7 & C_8 & D_9 & E_{10} \\
c_4 & b_5 & a_6 & \triangle_7 & A_8 & B_9 & C_{10} & D_{11} \\
d_5 & c_6 & b_7 & a_8 & \triangle_9 & A_{10} & B_{11} & C_{12} \\
e_6 & d_7 & c_8 & b_9 & a_{10} & \triangle_{11} & A_{12} & B_{13} \\
f_7 & e_8 & d_9 & c_{10} & b_{11} & a_{12} & \triangle_{13} & A_{14} \\
g_8 & f_9 & e_{10} & d_{11} & c_{12} & b_{13} & a_{14} & \triangle_{15}
\end{vmatrix}$$

We first find the corner solutions from the solutions for the 7-board, in which the diagonals are open. The only solution of this kind for the 7-board is, 2417536; and in this the second diagonal, B D, is open. Adding a fresh column to the board, along A D, and a fresh row along D C, we get a square at D on the 8-board, on which a queen may be placed; and we thus get the solution, 82417536. Also, adding a column along B C, and a row along A B, we get a square at B, on which a queen may be placed; and we thus get the solution 35286471. Each of these solutions has 8 aspects, which are to be found in the way explained above.



It may readily be ascertained without a diagram, whether any given solution has a diagonal open or not. Thus, taking the above solution for the 7-board, 2417536, we count from the left to the right, and find that 5 stands in the 5th place; therefore the first diagonal is closed. But, counting from right to left, we find that no number, r, in the solution, stands in the rth place; and therefore,

the second diagonal is open. Taking another solution for the 7-board, for instance, 2637415, we see that, counting from left to right, 3 stands in the 3rd place; and counting from right to left, 6 stands in the 6th place; both diagonals are therefore closed.

Having thus got all the corner solutions, we now, as explained by Glaisher, put zeros in place of the corner constituents Δ_1, Δ_{15}, G_8, g_8. This is equivalent to saying that in our systematic trials we need not begin, as Gauss did, with 1 in the 1st place; because this can only give us corner solutions that we have already found by a simpler process.

Proceeding now according to Glaisher's method, we have to develop the following:—

$$
\begin{array}{r|ccccccc|}
A_2 & a_2 & A_4 & B_5 & C_6 & D_7 & E_8 & F_9 \\
 & b_3 & \Delta_5 & A_6 & B_7 & C_8 & D_9 & E_{10} \\
 & c_4 & a_6 & \Delta_7 & A_8 & B_9 & C_{10} & D_{11} \\
 & d_5 & b_7 & a_8 & \Delta_9 & A_{10} & B_{11} & C_{12} \\
 & e_6 & c_8 & b_9 & a_{10} & \Delta_{11} & A_{12} & B_{13} \\
 & f_7 & d_9 & c_{10} & b_{11} & a_{12} & \Delta_{13} & A_{14} \\
 & . & e_{10} & d_{11} & c_{12} & b_{13} & a_{14} & . \\
\end{array} \quad - \quad - \quad (1)
$$

$$
\begin{array}{r|ccccccc|}
+B_3 & a_2 & \Delta_3 & B_5 & C_6 & D_7 & E_8 & F_9 \\
 & b_3 & a_4 & A_6 & B_7 & C_8 & D_9 & E_{10} \\
 & c_4 & b_5 & \Delta_7 & A_8 & B_9 & C_{10} & D_{11} \\
 & d_5 & c_6 & a_8 & \Delta_9 & A_{10} & B_{11} & C_{12} \\
 & e_6 & d_7 & b_9 & a_{10} & \Delta_{11} & A_{12} & B_{13} \\
 & f_7 & e_8 & c_{10} & b_{11} & a_{12} & \Delta_{13} & A_{14} \\
 & . & f_9 & d_{11} & c_{12} & b_{13} & a_{14} & . \\
\end{array} \quad - \quad - \quad (2)
$$

$$
\begin{array}{r|ccccccc|}
+C_4 & a_2 & \Delta_3 & A_4 & C_6 & D_7 & E_8 & F_9 \\
 & b_3 & a_4 & \Delta_5 & B_7 & C_8 & D_9 & E_{10} \\
 & c_4 & b_5 & a_6 & A_8 & B_9 & C_{10} & D_{11} \\
 & d_5 & c_6 & b_7 & \Delta_9 & A_{10} & B_{11} & C_{12} \\
 & e_6 & d_7 & c_8 & a_{10} & \Delta_{11} & a_{12} & B_{13} \\
 & f_7 & e_8 & d_9 & b_{11} & a_{12} & \Delta_{13} & A_{14} \\
 & . & f_9 & e_{10} & c_{12} & b_{13} & a_{14} & . \\
\end{array} \quad - \quad (3)
$$

+ determinants multiplied by D_5, E_6, and F_7.

Now, if we have found all the solutions containing A_2, we can

as Glaisher says, get those containing f_7, a_{14}, F_9, by turning the board through 1, 2, 3, right angles respectively; and those containing a_2, F_7, A_{14}, f_9, by reflection. We may, therefore, replace these 7 constituents by zeros in (2) and (3), and the subsequent determinants.

We have also in the first determinant, to replace by zeros all the constituents that contain A or 2; in the second determinant, all that contain B or 3; and so on; and we thus get the simplified expressions:—

$$
\begin{array}{rl}
A_2 & \left|\begin{array}{ccccccc}
. & . & B_5 & C_6 & D_7 & E_8 & F_9 \\
b_3 & \Delta_5 & . & B_7 & C_8 & D_9 & E_{10} \\
c_4 & a_6 & \Delta_7 & . & B_9 & C_{10} & D_{11} \\
d_5 & b_7 & a_8 & \Delta_9 & . & B_{11} & C_{12} \\
e_6 & c_8 & b_9 & a_{10} & \Delta_{11} & . & B_{13} \\
f_7 & d_9 & c_{10} & b_{11} & a_{12} & \Delta_{13} & . \\
. & e_{10} & d_{11} & c_{12} & b_{13} & a_{14} & .
\end{array}\right| \quad - \quad - \quad (4) \\
+B_3 & \left|\begin{array}{ccccccc}
. & . & . & C_6 & D_7 & E_8 & . \\
. & a_4 & A_6 & . & C_8 & D_9 & E_{10} \\
c_4 & b_5 & \Delta_7 & A_8 & . & C_{10} & D_{11} \\
d_5 & c_6 & a_8 & \Delta_9 & A_{10} & . & C_{12} \\
e_6 & d_7 & b_9 & a_{10} & \Delta_{11} & A_{12} & . \\
. & e_8 & c_{10} & b_{11} & a_{12} & \Delta_{13} & . \\
. & . & d_{11} & c_{12} & b_{13} & . &
\end{array}\right| \quad - \quad - \quad (5) \\
+C_4 & \left|\begin{array}{ccccccc}
. & \Delta_3 & . & . & D_7 & E_8 & . \\
b_3 & . & \Delta_5 & B_7 & . & D_9 & E_{10} \\
 & b_5 & a_6 & A_8 & B_9 & . & D_{11} \\
d_5 & c_6 & b_7 & \Delta_9 & A_{10} & B_{11} & . \\
e_6 & d_7 & c_8 & a_{10} & \Delta_{11} & A_{12} & B_{13} \\
. & e_8 & d_9 & b_{11} & a_{12} & \Delta_{13} & . \\
. & . & e_{10} & c_{12} & b_{13} & . & .
\end{array}\right| \quad - \quad - \quad (6)
\end{array}
$$

+determinants multiplied by D_5 and E_6.

Thus far we have followed the instructions given by Glaisher; but further simplifications are possible, which he does not mention, although he was probably aware of them. Glaisher calls a solution which begins with A_2, a penultimate solution; one that begins with B_3, an antepenultimate; and one that begins with C_4, a pre-

antepenultimate; but I prefer to call them second-place, third-place, and fourth-place solutions.

It is to be noticed that, when we have got the solutions that begin with A_2, B_3, or C_4, we can get those that begin with D_5, E_6, F_7, by the process I have termed reversion; and we may therefore reject the determinants multiplied by these quantities. Again, when we have got the solutions that begin with B_3, these will give us all the 3rd place solutions, so that we may in the determinant multiplied by C_4, put zeros for b_3, e_6, e_{10}, b_{13}, B_{13}, E_{10}. We thus get instead of (6) the following:—

$$\begin{vmatrix} C_4 & . & \Delta_3 & . & . & D_7 & E_8 & . \\ . & . & . & \Delta_5 & B_7 & . & D_9 & . \\ . & b_5 & a_6 & A_8 & B_9 & . & D_{11} & \\ d_5 & c_6 & b_7 & \Delta_9 & A_{10} & B_{11} & . & \\ . & d_7 & c_8 & a_{10} & \Delta_{11} & A_{12} & . & \\ . & e_8 & d_9 & b_{11} & a_{12} & \Delta_{13} & . & \\ . & . & . & c_{12} & . & . & . & \end{vmatrix} \quad - \quad - \quad (7)$$

Proceeding now to develop (4) we have first:

$$A_2B_5 \begin{vmatrix} b_3 & . & . & C_8 & D_9 & E_{10} \\ c_4 & a_6 & . & . & C_{10} & D_{11} \\ . & b_7 & \Delta_9 & . & . & C_{12} \\ e_6 & c_8 & a_{10} & \Delta_{11} & . & . \\ f_7 & d_9 & b_{11} & a_{12} & \Delta_{13} & . \\ . & e_{10} & c_{12} & b_{13} & a_{14} & . \end{vmatrix} \quad - \quad - \quad (8)$$

$$= A_2B_5b_3 \begin{vmatrix} a_6 & . & . & C_{10} & D_{11} \\ . & \Delta_9 & . & . & C_{12} \\ c_8 & a_{10} & \Delta_{11} & . & . \\ d_9 & . & a_{12} & \Delta_{13} & . \\ e_{10} & c_{12} & . & a_{14} & . \end{vmatrix} + A_2B_5C_8 \begin{vmatrix} c_4 & a_6 & . & . & D_{11} \\ . & b_7 & \Delta_9 & . & . \\ e_6 & . & a_{10} & . & . \\ f_7 & d_9 & b_{11} & \Delta_{13} & . \\ . & e_{10} & c_{12} & a_{14} & . \end{vmatrix}$$

$$+ A_2B_5D_9 \begin{vmatrix} c_4 & a_6 & . & . & . \\ . & b_7 & . & . & C_{12} \\ e_6 & c_8 & a_{10} & \Delta_{11} & . \\ f_7 & . & b_{11} & a_{12} & . \\ . & e_{10} & c_{12} & b_{13} & . \end{vmatrix} + A_2B_5E_{10} \begin{vmatrix} c_4 & a_6 & . & . & . \\ . & b_7 & \Delta_9 & . & . \\ e_6 & c_8 & . & \Delta_{11} & . \\ f_7 & d_9 & b_{11} & a_{12} & \Delta_{13} \\ . & . & c_{12} & b_{13} & a_{14} \end{vmatrix}$$

Here it may be useful to mention a practical hint that Glaisher gives, namely, that in forming the minor determinants it is convenient to write down the constituents not already obliterated, and then to scratch through with the pen, those that have to be replaced by zeros, in consequence of the new factor we have taken into the multiplier.

Proceeding now to develop the determinant multiplied by $A_2B_5b_3$, we have as the first minor determinant

$$A_2B_5b_3a_6 \left| \begin{matrix} \Delta_9 & . & . & C_{12} \\ . & \Delta_{11} & . & . \\ . & . & \Delta_{13} & . \\ c_{12} & . & . & . \end{matrix} \right|$$

and it is clear that this does not give a solution.
The second minor determinant is—

$$A_2B_5b_3C_{10} \left| \begin{matrix} . & \Delta_9 & . & . \\ c_8 & . & \Delta_{11} & . \\ d_9 & . & a_{12} & . \\ . & c_{12} & . & . \end{matrix} \right|$$

This vanishes, and of course gives no solution.
The third minor is—

$$A_2B_5b_3D_{11} \left| \begin{matrix} . & \Delta_9 & . & . \\ c_8 & a_{10} & . & . \\ d_9 & . & a_{12} & \Delta_{13} \\ e_{10} & c_{12} & . & a_{14} \end{matrix} \right|$$

As Δ_9 is the only constituent in the first row, and a_{12} the only one in the third column, the expanded determinant must contain both these; and we must then replace Δ_{13} and a_{14} by zeros, so that the fourth column contains only zeros, and the determinant vanishes.

We thus see that we cannot have a solution beginning with $A_2B_5b_3$, that is, with 241.

Next, taking the determinant multiplied by $A_2B_5C_8$, we see that, as D_{11} is the only constituent in the last column, it must be a factor in the expanded determinant, and we thus have to deal with the simpler determinant—

$$A_2B_3C_8D_{11}\begin{vmatrix} . & b_7 & \Delta_9 & . \\ e_6 & . & a_{10} & . \\ f_7 & d_9 & . & \Delta_{13} \\ . & e_{10} & c_{12} & a_{14} \end{vmatrix}$$

$$= A_2B_5C_8D_{11}b_7\begin{vmatrix} e_6 & a_{10} & . \\ . & . & \Delta_{13} \\ . & c_{12} & a_{14} \end{vmatrix} + A_2B_5C_8D_{11}\Delta_9\begin{vmatrix} e_6 & . & . \\ f_7 & . & . \\ . & e_{10} & a_{14} \end{vmatrix}$$

The first of these minor determinants gives the solution—

$$A_2B_5C_8D_{11}b_7e_6\Delta_{13}c_{12} \quad \text{or} \quad 24683175.$$

The second evidently can give no solution, and we thus find that this is the only solution beginning with 246.

Taking now the determinant multiplied by $A_2B_5D_9$, we see C_{12} is a factor of it, and we have to expand the simpler determinant

$$\begin{vmatrix} c_4 & a_6 & . & . \\ e_6 & c_8 & a_{10} & \Delta_{11} \\ f_7 & . & b_{11} & . \\ . & e_{10} & . & b_{13} \end{vmatrix} = c_4\begin{vmatrix} . & a_{10} & \Delta_{11} \\ . & b_{11} & . \\ e_{10} & . & b_{13} \end{vmatrix} + a_6\begin{vmatrix} . & . & \Delta_{11} \\ f_7 & b_{11} & . \\ . & . & b_{13} \end{vmatrix}$$

and it is obvious on inspection that neither of these gives a solution.

Lastly, taking the determinant multiplied by $A_2B_5E_{10}$, the first minor is—

$$A_2B_5E_{10}c_4\begin{vmatrix} b_7 & \Delta_9 & . & . \\ . & . & \Delta_{11} & . \\ d_9 & b_{11} & a_{12} & \Delta_{13} \\ . & . & b_{13} & a_{14} \end{vmatrix}$$

Here we see that Δ_{11} is a factor, since it is the only constituent in the second row; and as this involves replacing both Δ_9 and b_{11} by zeros, the determinant vanishes, and we get no solution.

The second minor is—

$$A_2B_5E_{10}a_6\begin{vmatrix} . & \Delta_9 & . & . \\ . & . & \Delta_{11} & . \\ f_7 & b_{11} & . & \Delta_{13} \\ . & c_{12} & b_{13} & . \end{vmatrix}$$

Here Δ_9 must be a factor, and this involves replacing Δ_{13} by zero, so that the determinant vanishes.

We thus finally conclude that there is only one solution which begins with A_2B_5 or with 24.

As a final example, I will take the beginning 257 or $A_2C_6D_9$. The determinant to be expanded in this case is—

$$\begin{vmatrix} c_4 & . & \Delta_7 & . & . \\ d_5 & b_7 & a_8 & . & . \\ . & c_8 & . & \Delta_{11} & B_{13} \\ f_7 & . & c_{10} & a_{12} & . \\ . & e_{10} & d_{11} & b_{13} & . \end{vmatrix} = B_{13} \begin{vmatrix} c_4 & . & \Delta_7 & . \\ d_5 & b_7 & a_8 & . \\ f_7 & . & c_{10} & a_{12} \\ . & e_{10} & d_{11} & . \end{vmatrix}$$

$$= B_{13}a_{12} \begin{vmatrix} c_4 & . & \Delta_7 \\ d_5 & b_7 & . \\ . & e_{10} & d_{11} \end{vmatrix}$$

This gives us the two products $c_4b_7d_{11}$, and $\Delta_7d_5e_{10}$; and arranging the constituents in the proper order, we get the solutions

$$A_2C_6D_9\, c_4\, b_7B_{13}a_{12}d_{11} \quad \text{or} \quad 25713864,$$

and $$A_2C_6D_9\Delta_7d_5B_{13}a_{12}e_{10} \quad \text{or} \quad 25741863.$$

On comparing the two processes—systematic trial and the use of determinants—we see that each step in the one process usually corresponds exactly to a step in the other.

When we replace Δ_1 by zero, this, as already remarkt, is equivalent to saying that we need not make trial of any combination that begins with 1. When we replace G_8 by zero, this is equivalent to saying that we are not to make trial of any combination that begins with 8; and when we similarly replace g_8 and Δ_{15}, this implies that we are not to try any combination that has 1 or 8 in the last place.

When in the determinant (2) (that is multiplied by B_3) we replace F_7 by zero, this implies that we are to try no combination that begins with 7. Replacing a_2 by zero implies that we are to try no combination that contains 1 in the 2nd place. Similarly replacing f_7 and A_{14} implies that we are not to try any combination in which either 1 or 8 stands in the 7th place; and, lastly, replacing f_9 and a_{14} by zero implies that we are not to try any combination in which 2 or 7 occupies the last place.

Proceeding next to the determinant (3) (that is multiplied by C_4) the directions as to replacing certain constituents by zeros imply that, in trying the combinations that begin with 4, we may neglect those in which either 1 or 8 stands in the 2nd or 3rd or in the 6th or 7th place; or in which 2, or 3, or 6, or 7, stands in the 8th place.

It is obvious, however, that very little labor is saved by excluding the combinations in which e_6, f_7, f_9, e_{10}, b_{13}, a_{14}, A_{14}, and B_{13} appear; and in practice it will be found convenient to try those combinations, so as to secure that we shall get at least two aspects of each solution.

The principal difference between the two processes is that, when determinants are used, each step in the process is recorded, so that any error can be easily traced and corrected.

Much time is occupied in writing down the determinants; but the process has the advantage of enabling us to take a comprehensive view of the combinations which we are to try. It sometimes also has the advantage of enabling us, when we are dealing with determinants containing 4 or 5 rows, to see more quickly than by the other method what combinations will give a solution. On the other hand, the method of systematic trial has the great advantage of furnishing a complete check on our work, because this can easily be so arranged that we shall obtain two (or more) aspects of each solution. Although, as already stated, I prefer the method of systematic trial, others may be of a different opinion; and each operator will probably prefer the method to which he is more accustomed.

We have seen that Glaisher gets the corner solutions for the n-board, from the solutions for the $(n-1)$-board in which the diagonals are open; and this has led Mr Rouse Ball to say that, "his method consists in deducing the solutions for a board of n^2 "cells, from one of $(n-1)^2$ cells". This statement, however, is very inaccurate; as there is no method, or at all events, none known at present, by which all the solutions for the n-board can be got from those of the $(n-1)$-board. The same erroneous idea appears again in the following passage: "The solutions for a board of 9^2 cells, "were given first by Prof. P. H. Schoute, of Holland, in the *Eigen* "*Haard*, and from them M. Delannoy constructed the solutions "for a board of 10^2 cells. The solutions are quoted by Lucas "(*Récréations*), Vol. II, pp. 238–240". I am not in a position to give any further information as to the methods followed by these gentlemen, the dates of their papers, or the results they have obtained.

As already mentioned, Pein has obtained the solutions for the 9-board, and for the 10-board; and the former agree exactly with those which I gave in my paper above referred to. I have since

obtained the solutions for the 11-board, and a list of them is appended to this paper. The corner solutions were got by means of Pein's solutions for the 10-board. I find there are:—

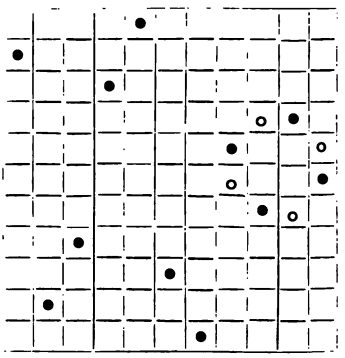
48	Corner solutions,	of	which	0	are	centric.
179	2nd place	„	„	4	„	„
82	3rd place	„	„	4	„	„
30	4th place	„	„	4	„	„
2	5th place	„	„	0	„	„
341	solutions in all	„	„	12	„	„

This was a very heavy piece of work, and occupied most of my leisure time for several months. The results thus obtained have settled a question I raised in my former paper. I there put forward the conjecture that it might be impossible that there should ever be four queens arranged in a solution, in the same position as they occupy in the solution on the 4-board; but this turns out not to be the case; for in the following solutions for the 11-board,

2.10.8.3.1.9.11.5.7.4.6,

and 2.10.8.3.1.9.11.6.4.7.5,

being Nos. 173 and 174 of the second-place solutions, the last 4 men in each case are so arranged. The two arrangements are shown in the annext figure.



I have also been able to settle a question raised by Pein. The solutions, when the number of sides is 4 and 5, are doubly symmetrical; each remaining unaltered when the board is rotated thro' a right angle. Pein says that such doubly symmetrical solutions seem to occur only when the number of squares in the side of the board is 4 or 5; but it will be found that the following solution for the 12-board is doubly symmetrical:—

5.3.11.6.12.9.4.1.7.2.10.8.

It is easy to see that a doubly symmetrical solution can only occur when the number of squares in a side is of the form $4N$ or $4N+1$.

The following are the numbers of solutions for the different boards:—

Number of Squares in a side.	Distinct Solutions.	Of which are centric.	Total Solutions.
4	1	1	2
5	2	1	10
6	1	1	4
7	6	2	40
8	12	1	92
9	46	4	352
10	92	3	724
11	341	12	2680

It will, I imagine, be scarcely practicable to obtain results for larger boards, unless a number of persons co-operate in the work. This would be very easy to arrange, if there were a sufficient number of volunteers; as the work is of such a nature that it could be readily divided among a number of persons.

Solutions for the Board containing 11 Squares in a side.

Note.—The solutions marked with a star (*) are centric.

(Here *t* is put for 10, and *e* for 11.)

Corner Solutions

No. 1.	13579 *e* 2468*t*	No. 16.	——*e*8 3 5926*t*
2.	13692 8 *e*475*t*	7.	14938 2 *e*6*t*75
3.	13794 2 *t*6*e*58	8.	——73 *t* 625*e*8
4.	——*t*2 9 64*e*85	9.	——*t* 2 *e*6835
5.	—— 8 *e* 24695	20.	14*t*7*e* 3 82596
6.	——*e*4 8 *t*5269	1.	1579*e* 3 8*t*246
7.	13869 2 5*e*47*t*	2.	——*t*4 *e* 93682
8.	——*t*4 9 *e*5726	3.	158*t*7 3 *e*2469
9.	——*e*7 2 *t*6495	4.	1596*t* 2 *e*3748
10.	13*t*79 *e* 24685	5.	164*t*8 3 52*e*97
1.	——84 *e* 96257	6.	——*e*8 5 2*t*379
2.	14692 *t* 538*e*7	7.	1683*e* 9 25*t*47
3.	——*t*3 *e* 85297	8.	——*t*4 9 52*e*37
4.	1479*e* 2 *t*6358	9.	——*e*7 4 2953*t*
5.	——*t*2 5 8*e*369	30.	16938 *e* 42*t*57

No. 31.	——7*t* 4 258*e*3	No. 40.	17*t*3*e* 9 42586
2.	16*e*5*t* 4 93827	1.	——83 – ——*e*6
3.	1748*e* 5 *t*2639	2.	18497 3 *e*625*t*
4.	1753*e* 9 42*t*86	3.	——*e*– – *t*—95
5.	——8*e* 4 *t*3692	4.	185*e*6 3 *t*4792
6.	——— 9 42*t*36	5.	18639 *e* 4*t*752
7.	17938 *t* 425*e*6	6.	——*e*3 5 94*t*72
8.	——63 *t* 4*e*852	7.	1946*e* 3 *t*7582
9.	——*e*3 5 *t*2468	8.	1963*e* 4 8*t*572

Second-Place Solutions.

No. 1.	2417*t* 3 *e*8596	No. 29.	——*t* 3 58196
2.	24693 1 *e*75*t*8	30.	—–95 1 *t*7368
3.	——– – –8–7*t*	1.	25184 *e* 936*t*7
4.	24718 *e* 5396*t*	2.	——9*e* 8 473*t*6
5.	—–9*e* 5 *t*1683	3.	—–*e*7 *t* 63948
6.	*——— 6 1358*t*	4.	2531*t* 8 *e*4796
7.	—–*t*3 *e* 61958	5.	2579*e* 3 168*t*4
8.	——*e*– 5 *t*—6–	6.	——— 8 413*t*6
9.	24819 6 3*e*75*t*	7.	——*t*3 *e* 64918
10.	——*e* 6 *t*7539	8.	——4 *e* 13968
1.	——*t*3 1 6*e*975	9.	——— – –8639
2.	——5 – —3–9	40.	——*e*1 6 4*t*839
3.	——– – *e*7369	1.	——3 8 4196*t*
4.	—–*e*3 6 *t*5197	2.	25813 *t* 74*e*96
5.	——- *t* 7——6	3.	——*e* 4 7*t*396
6.	——5 1 *t*6397	4.	—–*e*1 – ———
7.	——— *t* 61379	5.	*——3 6 9147*t*
8.	24973 *e* 615*t*8	6.	——— *t* 74196
9.	——*t* 1 *e*5863	7.	——7 *t* 31946
20.	——– – –6–35	8.	——9 3 147*t*6
1.	—–*e*5 *t* 16837	9.	——— 4 613*t*7
2.	——8 5 317*t*6	50.	*——— 6 3147*t*
3.	24*t*39 *e* 58617	1.	25964 *e* 137*t*8
4.	——79 *e* 31685	2.	25*t*6*e* 3 74819
5.	——*e* 6 31859	3.	——83 *e* 91647
6.	——83 *e* 71695	4.	——4 *e* 73169
7.	——5 *e* 13697	5.	——*e* 4 13697
8.	24*e*73 *t* 61958	6.	25*e*6*t* 3 18497

No. 57. ——91 *t* 47386
8. ——4 6 3*t*718
9. 2639*e* 4 *t*7518
60. 2683*e* 9 514*t*7
1. —*t*4 1 3*e*975
2. ——5 1 9*e*374
3. —*e*1 3 7*t*495
4. ——4 1 953*t*7
5. ——5 1 *t*4793
6. ——7 1 3594*t*
7. 2691*t* 4 7*e*835
8. —*e*1 — —*t*—
9. 26*t*14 9 73*e*85
70. ——*e* 9 53847
1. —31 8 5*e*974
2. ——7 *e* 48159
3. —71 3 58*e*94
4. 26*e*37 4 *t*1958
5. 2716*e* 5 *t*4938
6. —8*e* 9 4*t*536
7. —*e*9 6 4*t*835
8. 2738*e* 4 1*t*596
9. —*t*8 1 *e*4695
80. —*e*8 4 1*t*596
1. ——9 5 1*t*468
2. 27531 9 *e*64*t*8
3. —8*e* 1 4693*t*
4. —— 9 3641*t*
5. ——— — 4–13–
6. 275*t*1 4 *e*8396
7. —— 6 4*e*839
8. ——8 6 *e*3194
9. —*e*1 *t* 63948
90. 27915 *t* 63*e*84
1. —41 5 *t*6*e*38
2. —61 *t* 4*e*853
3. ——3 1 —*t*5
4. ——— — *e*5—4
5. —*e*5 1 4*t*863

No. 96. 27*t*31 9 58*e*46
7. ——5 8 *e*4619
8. ——— 9 14*e*86
9. ——9 6 4*e*185
100. —41 5 9*e*386
1. ——— - —6–3
2. ——— 8 *e*3695
3. ——9 1 3*e*685
4. —63 1 *e*8594
5. ——— 9 *e*4815
6. —83 5 9*e*146
7. 27*e*35 9 1*t*468
8. ——8 6 4*t*159
9. —48 1 3*t*695
110. —81 4 *t*3——
1. ——3 1 *t*5964
2. 28394 *t* 16*e*57
3. ——— – 5*e*61–
4. —*e*7 5 *t*1469
5. ——— *t* 41596
6. 28519 *e* 637*t*4
7. —3*t* 6 *e*1479
8. —7— — —93
9. *—91 6 *e*374*t*
120. ——4 *t* 7*e*316
1. 28613 *e* 7*t*495
2. —39 *e* 157*t*4
3. —4*t* 1 *e*5793
4. —91 *t* 53*e*74
5. 28*t*14 *e* 96357
6. —39 6 41*e*57
7. —49 3 57*e*16
8. —73 1 *e*4695
9. 29184 *e* 7*t*635
130. 29317 *t* 6*e*584
1. ——8 *t* 57*e*46
2. —6*t* 1 5*e*847
3. —8*t* 4 61*e*75
4. ——*e* 1 6*t*574

No. 135. ——*t*7 1 *e*5864
6. 29514 *t* 7*e*386
7. ——8 4 *e*73*t*6
8. ——*e* 6 *t*7483
9. ——3*t* 6 *e*1748
140. ——81 3 6*e*7*t*4
1. ——— 4 7*e*63*t*
2. ——*t* 3 6*e*147
3. 29613 8 *e*74*t*5
4. ——7 4 *e*853*t*
5. ——31 8 *e*57*t*4
6. ——— *t* 7–*e*8–
7. ——*t* 8 *e*4175
8. ——41 *t* 57*e*38
9. ——*t*5 1 *e*4738
150. ——— *e* 1——
1. 29731 6 *t*5*e*84
2. ——41 *e* 6*t*358
3. ——*t* 1 5*e*863
4. 29*e*61 3 *t*7485
5. 2*t*374 8 *e*5169
6. ——— *e* 96158
7. ——9 4 61*e*58

No. 158. 2*t*571 4 *e*8693
9. ——4 1 ——39
160. 2*t*615 8 *e*3794
1. ——— *e* 97483
2. ——9 *e* 3——5
3. ——39 4 18*e*57
4. ——— *e* 4–1—
5. ——49 1 37*e*85
6. ——93 1 *e*8574
7. ——— 5 —1—
8. ——5 1 47*e*83
9. ——*e*3 1 95847
170. 2*t*73*e* 9 41586
1. ——41 5 9*e*683
2. ——— 9 63*e*85
3. 2*t*831 9 *e*5746
4. ——— – –6475
5. ——5 – –1746
6. ——49 3 6*e*157
7. ——63 9 *e*1475
8. ——*e*4 1 36975
9. ——5 3 16497

Third-Place Solutions.

No. 1. 3528*e* 4 196*t*7
2. 3579*e* 2 41*t*86
3. ——*t*4 *e* 18629
4. ——6 – —4—
5. 3581*e* 6 27*t*49
6. *——*t*1 6 *e*2479
7. ——— – —794
8. ——*e*9 4 61*t*27
9. 35914 *e* 862*t*7
10. ——*t* 7 *e*8246
1. ——26 *e* 174*t*8
2. ——*t* 7 *e*8146
3. ——*e*4 7 162*t*8
4. 35*t*49 *e* 27168

No. 15. ——86 – ——49
6. 362*t*1 4 8*e*975
7. 3641*e* 9 528*t*7
8. 3681*e* 7 *t*2594
9. ——— 9 524*t*7
20. ——*t*1 4 2*e*975
1. ——*e*2 7 *t*4195
2. ——4 1 75*t*29
3. 36924 *e* 851*t*7
4. ——8 *e* 41*t*57
5. ——— – –71*t*5
6. ——71 *e* 25*t*84
7. ——2 *e* 814*t*5
8. ——*e*2 5 *t*8147

No. 29.	36*t*24 1 8*e*975	No 56.	——6 1 *e*7529
30.	——72 *e* 58194	7.	*——51 6 *e*7249
1.	——— – 84—5	8.	——74 1 *e*9625
2.	36*e*14 7 *t*8259	9.	38*e*52 9 147*t*6
3.	——8 5 297*t*4	60.	——92 5 ———
4.	——51 4 *t*7928	1.	39248 1 *e*6*t*75
5.	——91 5 28*t*74	2.	——*e* 7 *t*6158
6.	3748*e* 9 152*t*6	3.	——*t*1 6 8*e*475
7.	——*t*1 5 *e*8629	4.	394*t*1 6 *e*2758
8.	*——— 6 –285–	5.	——— *e* 6——
9.	——*e*1 5 *t*6928	6.	——5 – –1–2–
40.	37915 *t* 8*e*246	7.	3961*e* 7 528*t*4
1.	——8 5 *e*2*t*64	8.	——*t*5 *e* 17428
2.	——26 1 *e*58*t*4	9.	39742 *e* 61*t*85
3.	—*t*— – –859–	70.	3*t*296 1 *e*8574
4.	——41 9 *e*6825	1.	3*t*419 *e* 68257
5.	——82 5 1*e*964	2.	——28 *e* 71695
6.	37*e*81 4 692*t*5	3.	——71 *e* 62958
7.	38241 9 *e*6*t*75	4.	——*e* 2 61—–
8.	*——71 6 *e*5*t*49	5.	3*t*64*e* 1 85297
9.	——9 *e* 15*t*64	6.	——95 1 *e*7248
50.	——96 *t* 1*e*574	7.	3*t*716 *e* 28594
1.	38429 5 1*e*6*t*7	8.	——2– – 1——
2.	——*e*1 *t* 75296	9.	——*e* 5 81946
3.	3861*t* 7 *e*4295	80.	——41 *e* 86295
4.	——*e*2 5 194*t*7	1.	3*t*842 7 *e*1596
5.	38*t*41 9 *e*6275	2.	——5*e* 1 62974

Fourth-Place Solutions.

No. 1.	427*t*6 *e* 91358	No. 11.	46*t*25 *e* 81397
2.	428*t*3 7 *e*1695	2.	——7 *e* 31958
3.	——7 1 –9–3–	3.	——5*e* 1 37928
4.	42936 *e* 1*t*758	4.	47931 8 *e*52*t*6
5.	——*e*1 8 63*t*75	5.	——*e*2 5 18*t*36
6.	——3 – –1——	6.	*47*t*31 6 *e*9258
7.	42*t*5*e* 8 13697	7.	——— 8 —625
8.	46931 *e* 7*t*258	8.	*——*e* 6 19258
9.	——*e*1 5 2*t*738	9.	——63 1 *e*9528
10.	——2 7 318*t*5	20.	483*e*2 7 1*t*596

No. 21.	——9 1 6t275	No 26.	49382 e 16t57
2.	—— 5 1t726	7.	*495t1 6 e2738
3.	48591 3 e72t6	8.	*4972e 6 1t538
4.	—— - —t2-	9.	——3e 8 15t26
5.	48t59 e 13627	30.	4t36e 1 85297

FIFTH-PLACE SOLUTIONS.

No. 1.	524t8 e 13697	No. 2.	536t2 e 18497

On a Problem of Lewis Carroll's.

By Professor STEGGALL.

Fifth Meeting, 10*th March*, 1899.

ALEXANDER MORGAN, Esq., M.A., D.Sc., President, in the Chair.

Centrobaric Spherical Surface Distribution.

By Professor TAIT.

The following is a simple geometrical demonstration of the well-known theorem that, if matter be distributed over a sphere with a surface-density (*i.e.*, mass per unit of surface) inversely as the cube of the distance from either of two points which are the inversions of each other with respect to the sphere, it will act upon all external masses as if it were collected at the interior point:—and upon all internal masses as if a definite multiple of its mass were concentrated at the exterior point.

Suppose a cone of very small angle, whose vertex is S, to cut out small areas, P and Q, from a spherical surface. (Fig 5.) Then we have, obviously,

$$\frac{P}{SP^2} = \frac{Q}{SQ^2}$$

And, of course, the rectangle SP.SQ is constant, say c^2.

Let R be any point, outside the sphere if S be inside, and *vice versâ*; and take T (always *inside* the sphere) on RS so that $SR.ST = c^2$.

Then, by similar triangles, we have

$$SQ.PR = SR.QT.$$

From these it follows directly that

$$\frac{P}{SP^3}\Big/PR = \frac{Q}{SQ^2.SP.PR} = \frac{Q}{c^2SR.QT}.$$

Thus

$$\Sigma\left(\frac{P}{SP^3}\cdot\frac{1}{PR}\right) = \frac{1}{c^2SR}\Sigma\left(\frac{Q}{QT}\right).$$

The first member is the potential, at R, if the surface-density be everywhere inversely as the cube of the distance from S. The second is the potential, at R, of a mass *concentrated at S;* since $\Sigma(Q/QT)$ is constant, being the potential of the uniform shell at an internal point.

The mass of the centrobaric shell is

$$M = \Sigma\frac{P}{SP^3} = \Sigma\frac{Q}{SQ^2.SP} = \frac{1}{c^2}\Sigma\frac{Q}{SQ}$$

so that the expression for its potential at R is

$$\frac{\Sigma\frac{Q}{QT}}{\Sigma\frac{Q}{QS}}\cdot\frac{M}{SR}.$$

While S is inside the shell, the first factor is unity; otherwise it is directly as the ratio of the distance of S from the centre of the sphere, to the radius. Thus we prove by elementary considerations the important propositions enunciated above.

On Wireless Telegraphy and High Potential Currents.

By J. R. Burgess, M.A.

Sixth Meeting, 12th May 1899.

Alexander Morgan, Esq., M.A., D.Sc., President, in the Chair.

Discussion on "The Treatment of Proportion in Elementary Mathematics."

Seventh Meeting, 9th June 1899.

ALEXANDER MORGAN, Esq., M.A., D.Sc., President, in the Chair.

Systems of Circles analogous to Tucker Circles.*

By J. A. THIRD, M.A.

I.

SYSTEMS OF SIX-POINT CIRCLES CONNECTED WITH THE TRIANGLE. GENERAL THEOREMS.†

1. If a circle meets the sides BC, CA, AB of a triangle (Fig. 6) in the pairs of points L and l, M and m, N and n respectively, it is obvious that the pairs of connectors Mn and mN, Nl and nL, Lm and lM are antiparallel with respect to the angles A, B, C respectively.

Conversely, by a modification of a theorem of Poncelet's, if the above-mentioned pairs are antiparallel with respect to A, B, C respectively, the six points L, l, M, m, N, n lie on a circle; for if not, M, n, N, m and N, l, L, n and L, m, M, l would lie on three distinct circles, and consequently BC, CA, AB, their radical axes, would be concurrent.

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* Mr Third's Papers were received on 29th March. Sections I. and II. of the First Paper, in a different form, were read at the November and December Meetings.

† Since writing this paper I have found that a considerable number of the theorems contained in Part I. are given in two English memoirs, which I had somehow overlooked, viz., "The Relations of the Intersections of a Circle with a Triangle," by H. M. Taylor, *Proc. of the London Math. Soc.*, Vol. XV., pp. 122-139, in which the treatment is mainly analytical, and "Some Geometrical Proofs of Theorems connected with the Inscription of a Triangle of constant Form in a given Triangle," by M. Jenkins, *Quarterly Journal*, Vol. XXI., pp. 84-89. Among the theorems in Part I. which seem to be new, the most important, perhaps, are those referring to the connection between the S-point and the angles of a system, and to the relation of the systems to coaxal systems. The special cases of Arts. 19 and 22, Part II., are also referred to in the above-mentioned papers, but in the former of these cases the S-point is not determined.

2. It follows at once from Pascal's theorem that in the figure of the preceding article (Fig. 6) the following are triads of collinear points

A, and the points of intersection of the pairs Lm, Nl and lM, nL;
B, „ „ „ „ „ „ „ „ Mn, Lm and mN, lM;
C, „ „ „ „ „ „ „ „ Nl, Mn and nL, mN;

and also that the three lines on which these triads of points lie are concurrent. Let the point of concurrence be denoted by S.

It is obvious that by joining the six points in which the circle meets the sides of the triangle in all possible ways by chords passing from side to side, we obtain, in general, three other points of concurrence similar to S.

3. If three lines be drawn parallel to Mn, Nl, Lm (Fig. 6) meeting CA, AB in M′, n'; AB, BC in N′, l'; BC, CA in L′, m' respectively, and intersecting two and two on SC, SA, SB, the six points L′, l', M′, m', N′, n' also lie on a circle.

Demonstration. Let Lm, Nl intersect in P, and L′m', N′l' in P′. Then $AN/AN' = AP/AP' = Am/Am'$. Therefore mN, m'N′ are parallel. Therefore M′n', m'N′ are antiparallel with respect to the angle A. Similarly it may be shown that N′l', n'L′ are antiparallel with respect to the angle B, and that L′m', l'M′ are antiparallel with respect to the angle C. Therefore, by (1), L′, l', M′, m', N′, n' are concyclic.

By drawing a system of parallels to Mn, Nl, Lm as above, we obtain a system of six-point circles of which the original circle is one. Such a system may be called, in the meantime, an S-point system. The chords Mn, Nl, Lm of the circle which is regarded as the original circle of the system may be called the directive chords of the system.

It is obvious that the same S-point system would be obtained by drawing parallels (intersecting two and two on SC, SA, SB) to mN, nL, lM, *i.e.*, by drawing antiparallels to Mn, Nl, Lm with respect to the angles A, B, C respectively. In other words mN, nL, lM may equally well with Mn, Nl, Lm be regarded as directive chords of the system.

The following particular cases may be noticed.

(i) Two circles of the system are, in general, obtained by drawing (1) parallels and (2) antiparallels to the directive chords, so as to cointersect in S.

(ii) Two circles of the system are, in general, determined by drawing through any point P′ on SA (or SB or SC) (1) parallels and (2) antiparallels to the two directive chords which meet on that line; but in the case where P′ also lies on BC (or CA or AB), these two circles coalesce and touch BC (or CA or AB) at P′.

(iii) If from A, lines AX, AX′ be drawn parallel to the directive chords which intersect on SA, to meet the opposite side BC, and if from B and C analogous pairs of lines BY, BY′ and CZ, CZ′ be drawn to meet the opposite sides, the circumcircles of the triangles AXX′, BYY′ and CZZ′ belong to the system.

It is obvious that if we take in succession, as S-point, each of the three other points of concurrence referred to in Art. 2 as being obtainable in connection with the original circle, we obtain three other S-point systems. Thus, in general, every six-point circle connected with a triangle, belongs to four different S-point systems. It will be noticed that while each circle of a system has, in general, four S-points, the circles of the system have one and only one S-point in common.

4. If x, y, z be the trilinear coordinates of S, and α, β, γ respectively denote the angles AnM, BlN, CmL which the directive chords make with the sides of ABC, (Fig. 6) we have the following relations:

$$\text{(i)} \quad x/\operatorname{cosec}\alpha \operatorname{cosec}(A+\alpha) = y/\operatorname{cosec}\beta \operatorname{cosec}(B+\beta) = z/\operatorname{cosec}\gamma \operatorname{cosec}(C+\gamma);$$

$$\text{and} \quad \text{(ii)} \quad \alpha+\beta+\gamma=\pi.$$

Demonstration. Let M″n″, N″l″, L″m″ be drawn through S parallel to Mn, Nl, Lm respectively, and meeting the sides of ABC as in the figure. Then, by Art. 3 (i), L″, l″, M″, m″, N″, n″ lie on a circle of the system. Therefore

$$M''S \cdot Sn'' = N''S \cdot Sl'' = L''S \cdot Sm''.$$

But $M''S = y\cos ec\, A\widehat{M}''S = y\cos ec(A+\alpha)$,

and $Sn'' = z\cos ec\, A\widehat{n}''S = z\cos ec\alpha$.

Therefore $M''S \cdot Sn'' = yz\cos ec\alpha\cos ec(A+\alpha)$.

Similarly it may be shown that

$$N''S \cdot Sl'' = zx\cos ec\beta\cos ec(B+\beta),$$

and $L''S \cdot Sm'' = xy\cos ec\gamma\cos ec(C+\gamma)$.

Hence the first part of the proposition is established.

Again $B\widehat{n}L = B\widehat{l}N = \beta$,

and $M\widehat{n}L = C\widehat{m}L = \gamma$.

Therefore, the signs being suitably selected,

$$\alpha+\beta+\gamma = \pi.$$

The angles α, β, γ (or $\pi - A - \alpha$, $\pi - B - \beta$, $\pi - C - \gamma$) which the directive chords of an S-point system (or their antiparallels) make with the sides of ABC, may be called the angles of the system. The theorem established in the present article enables us to state that the angles of an S-point system and the position of S implicate each other, and that if either of these data be given, the system is determinate. The theorem is obviously calculated to be of great service in the investigation of particular cases of S-point systems connected with the triangle. The discussion of these, however, is reserved for Part II.

It may be noted that $LL'/L'L'' = ll'/l'l'' = MM'/M'M'' = \text{etc.}$, from which it could readily be proved that the centres of the circles of an S-point system are collinear; but this will be established by a shorter method in Art. 8.

5. If α, β, γ be the angles of an S-point system of circles connected with the triangle ABC, and if on the sides of ABC (Fig. 7) there be described externally three similar triangles aBC, AbC, ABc, such that

$$C\widehat{a}B = C\widehat{A}b = c\widehat{A}B = \alpha,$$

$$a\widehat{B}C = A\widehat{b}C = A\widehat{B}c = \beta,$$

and $$B\widehat{C}a = b\widehat{C}A = B\widehat{c}A = \gamma,$$

then Aa, Bb, Cc cointersect in a point,* which I shall denote by II,

* This construction is given by Mr Jenkins in the paper already cited.

whose trilinear coordinates are

$$\sin\alpha\cos ec(A+\alpha),\ \sin\beta\cos ec(B+\beta),\ \sin\gamma\cos ec(C+\gamma).$$

Also if other three similar triangles $a'BC$, $Ab'C$, ABc' be described externally on the sides of ABC, such that

$$C\widehat{a'}B = C\widehat{A}b' = c'\widehat{A}B = \pi - (A+\alpha),$$
$$a'\widehat{B}C = A\widehat{b'}C = A\widehat{B}c' = \pi - (B+\beta),$$
and $$B\widehat{C}a' = b'\widehat{C}A = B\widehat{c'}A = \pi - (C+\gamma),$$

then Aa', Bb', Cc' cointersect in a point, which I shall denote by Π', whose trilinear coordinates are

$$\cos ec\alpha\sin(A+\alpha),\ \cos ec\beta\sin(B+\beta),\ \cos ec\gamma\sin(C+\gamma).$$

The points Π and Π' are obviously isogonal conjugates with respect to the triangle ABC.

They will be referred to subsequently as the Π-points of the S-point system of circles whose angles are α, β, γ. It will be found that the Π-points of an S-point system play an analogous part to that played by the Brocard points, which are a particular case of them, in the theory of the Tucker circles.

Other methods of constructing the Π-points of a system might easily be deduced from the following considerations.

$$\Pi aBC,\ \Pi Ab C,\ \Pi ABc,\ \Pi' a'BC,\ \Pi' Ab'C,\ \Pi' ABc'$$

are evidently quartets of concyclic points.

Hence $B\widehat{\Pi}C = \pi - \alpha$, etc., $\Pi\widehat{B}C + \Pi\widehat{C}B = \alpha$, etc.,

$B\widehat{\Pi'}C = A + \alpha$, etc., $\Pi'\widehat{B}C + \Pi'\widehat{C}B = \pi - (A+\alpha)$, etc.

6. The pedal circle of the Π-points of an S-point system of circles connected with the triangle ABC, is one of the circles of the system.

Demonstration. Let l, m, n (Fig 8) be the feet of the perpendiculars from Π on BC, CA, AB respectively, and L, M, N the feet of the corresponding perpendiculars from Π'. Then, by a well-known theorem of Steiner's, since Π and Π' are isogonal conjugates L, l, M, m, N, n, lie on a circle. Hence, making the necessary joins and noticing that $BL\Pi'N$ and $CM\Pi'L$ are cyclic quadrilateral we have

$$A\widehat{n}M = N\widehat{L}M = \Pi'\widehat{B}A + \Pi'\widehat{C}A = \Pi\widehat{B}C + \Pi\widehat{C}B = \alpha.$$

Similarly it may be shown that $B\widehat{l}N = \beta$, and $C\widehat{m}L = \gamma$. Therefore the circle $L l M m N n$ belongs to the S-point system whose angles are α, β, γ, *i.e.*, to the system whose Π-points are Π and Π'.

It will be noticed that the centre of the circle $L l M m N n$ is the mid-point of $\Pi\Pi'$.

It is evident that an S-point system connected with the triangle is completely determined, if its Π-points are given.

7. If the pedal triangles of Π and Π' be rotated round Π and Π' respectively, in opposite directions through equal angles, so as to remain inscribed in ABC and retain their species, they have a common circumcircle which belongs to the S-point system of which Π and Π' are the Π-points.

Demonstration. Let Πl, Πm, Πn (Fig. 8) be rotated round Π in, say, the positive direction through an angle ϕ, and in their new positions let them meet the sides of ABC in l', m', n'. Similarly let $\Pi'L$, $\Pi'M$, $\Pi'N$ be rotated round Π' in the negative direction through an equal angle ϕ, meeting in their new positions the sides of ABC in L', M', N'. Then, as might easily be proved, the triangles lmn and LMN remain constant in species during the above operation; and $L'l'M'm'$, $M'm'N'n'$, $N'n'L'l'$ are cyclic quartets, whence by Art. 1, L', l', M', m', N', n' are concyclic.

Again, making the necessary joins, and denoting the angles ΠAB, $\Pi'AC$ by θ, we have

$$MM' = \Pi'M \tan\phi = AM \tan\theta \tan\phi,$$

and

$$nn' = \Pi n \tan\phi = An \tan\theta \tan\phi.$$

Therefore $M'n'$ is parallel to Mn. Hence $A\widehat{n'}M' = \alpha$.
Similarly it may be shown that $B\widehat{l'}N' = \beta$, and $C\widehat{m'}L' = \gamma$. Therefore the circle $L'l'M'm'N'n'$ belongs to the S-point system whose angles are α, β, γ, *i.e.* to the system whose Π-points are Π and Π'.

It will be noticed that

$$\widehat{nlm} = \Pi\widehat{BA} + \Pi\widehat{CA} = \Pi'\widehat{BC} + \Pi'\widehat{CB} = \pi - (A + \alpha), \text{ etc.},$$

and

$$\widehat{NLM} = \Pi'\widehat{BA} + \Pi'\widehat{CA} = \Pi\widehat{BC} + \Pi\widehat{CB} = \alpha, \text{ etc.}$$

Hence the angles of the variable triangle $(l'm'n')$ associated with Π, are equal to the supplements of $A + \alpha$, $B + \beta$, $C + \gamma$, while those of the variable triangle (L'M'N') associated with Π', are equal to α, β, γ.

An interesting particular case not amongst those discussed in Part II. may be referred to here. When Π is on the circumcircle of ABC, Π′, its isogonal conjugate, is at infinity, and the pedal triangle of Π degenerates into the Simson line of Π. In this case the circles of the S-point system degenerate into a system of line-pairs, one line of each pair being the line at infinity and the other lines the Simson lines of Π (the lines joining the feet of isoclinals from Π to the sides of ABC being included in this designation).

8. The locus of the centres of an S-point system of circles is the straight line which bisects the connector of the Π-points perpendicularly.

Demonstration. Let K, K′ (Fig. 8) be the centres of the circles L*l*M*m*N*n* and L′*l*′M′*m*′N′*n*′ respectively, and join KK′, K′Π, K′Π′. Then K is the mid-point of ΠΠ′, and the angles KΠK′, KΠ′K′ are obviously each equal to the angle of rotation ϕ. Therefore KK′ bisects ΠΠ′ perpendicularly.

When L*m*, L′*m*′ etc., are equally inclined to BC, CA and N*l*, N′*l*′, etc., to AB, BC, it is obvious that the perpendicular bisectors of L*m*, L′*m*′, etc., coalesce, as also those of N*l*, N′*l*′, etc.; so that in this case the line of centres reduces to a point.

9. The radii of the circles of an S-point system bear a constant ratio to the distances of their centres from either of the Π-points of the system.

Demonstration. If K*n*, K′*n*′ (Fig. 8) be joined, it is obvious that the triangles ΠK*n*, ΠK′*n*′ are similar. Hence K′*n*′/K*n* = K′Π/KΠ, which proves the proposition.

From this property it is evident that if we construct a system of coaxal circles having Π and Π′ as common points, the circles of the S-point system are simply those of the coaxal system with their radii increased or diminished in a constant ratio, the centres remaining fixed. I propose to call a system of circles arrived at in this manner, by the elongation or shortening in a constant ratio of the radii of a coaxal system whether of the common-point or limiting-point species, a coaxaloid system of circles. The ratio in which the radii of the kernel coaxal system are increased or diminished will be referred to as the ratio of the resulting coaxaloid system. It is

evident that coaxaloid systems of circles could be studied altogether apart from their connection with the triangle, and in fact many of their properties can be so obtained, of which the theorems of Arts. 14 to 17 are examples. It is evident also that while every S-point system as previously defined, is a coaxaloid system, it does not follow that every coaxaloid system is an S-point system of a triangle with which its Π-points are connected as isogonal conjugates. In fact for every pair of real Π-points, connected with a triangle as isogonal conjugates, we have an infinite number of coaxaloid systems with different ratios, but only one of these systems with a definite ratio depending on the position of the Π-points, is a system of six-point circles of the triangle, possessing the property of parallel chords. When the Π-points are imaginary, *i.e.* when the coaxaloid systems are derived from a limiting-point instead of a common-point coaxal system, none of the coaxaloid systems are S-point systems. In future an S-point system of circles will be called a coaxaloid system connected with the triangle.

10. Π and Π′ being the Π-points of a coaxaloid system of circles connected with the triangle, we will now consider the connection between the magnitude of the ratio of the system, denoted by r, and the position of Π and Π′. Let d denote the diameter of the pedal circle of Π and Π′. Then $d/\Pi\Pi' = r$.

(i) When Π and Π′ both lie within the triangle, $d > \Pi\Pi'$ and $r > 1$.

(ii) When Π and Π′ lie on the perimeter of the triangle, in which case one of them coincides with a vertex and the other lies on the opposite side, $d = \Pi\Pi'$, and $r = 1$. In this case the coaxaloid system connected with the triangle is an ordinary common-point coaxal system.

(iii) When one of the Π-points lies outside the triangle but within the circumcircle, in which case the other lies outside the circumcircle, $d < \Pi\Pi'$, and $r < 1$.

(iv) When one of the Π-points lies on the circumcircle, in which case the other is at infinity (see Art. 8), $d = \Pi\Pi' = \infty$, and $r = 1$.

(v) When Π and Π′ both lie outside the circumcircle, $d > \Pi\Pi'$ and $r > 1$.

(vi) When Π and Π' coincide, as they do in the incentre and excentres, $\Pi\Pi' = 0$ and $r = \infty$.

(vii) If Kl (Fig. 8) be joined it may easily be proved that $4Kl^2 = \Pi\Pi'^2 + 4\Pi l . \Pi'L$. But Kl is a radius of the pedal circle of Π and Π'. Hence $r^2 = 1 + 4\Pi l . \Pi'L/\Pi\Pi'^2$. This result supplies us with a ready means of calculating the ratio in certain cases.

11. The envelope of a coaxaloid system of circles connected with a triangle ABC is the conic inscribed in (or escribed to) the triangle, and having the Π-points of the system as foci.

Demonstration. Let K be the mid-point of $\Pi\Pi'$ (Fig. 9), and TT' the diameter through Π and Π' of the pedal circle of Π and Π'. Let K' be the centre of any other circle of the system, and let this circle be cut by the circumcircle of $\Pi\Pi'K'$ at J. Join $J\Pi$, $J\Pi'$, JK', $\Pi K'$ and $\Pi'K'$.

(i) Suppose $r > 1$, and consequently $KT > K\Pi$. Then by Ptolemy's theorem

$$J\Pi . K'\Pi' + J\Pi' . K'\Pi = \Pi\Pi' . K'J.$$

But $K'\Pi' = K'\Pi$.

Therefore
$$J\Pi + J\Pi' = \Pi\Pi' . K'J/K'\Pi$$
$$= \Pi\Pi' . KT/K\Pi = TT'.$$

Therefore J lies on an ellipse having Π and Π' as foci, and the pedal circle of Π and Π' as auxiliary circle.

Again, since $K'\Pi' = K'\Pi$, JK' bisects the angle $\Pi J\Pi'$, and is, consequently, the normal to the ellipse at J. Therefore the circle K' touches the ellipse at J. Similarly it might be proved to touch the ellipse at the image of J with respect to the line of centres of the coaxaloid system.

Again, since the feet of the perpendiculars from Π and Π' on the sides of ABC, lie on the pedal circle of Π and Π', *i.e.* on the auxiliary circle of the ellipse, the sides of ABC are tangents to the ellipse.

It may be noted that when Π and Π' coincide, the ellipse becomes the incircle or one of the excircles.

(ii) When $r < 1$, it may be proved in precisely the same way that the envelope is a hyperbola touching the sides of ABC and having Π, Π' as foci and their pedal circle as auxiliary circle.

(iii) In the particular case when one of the II-points lies on the circumcircle, the coaxaloid system degenerates, as we have seen, into the Simson lines of the point, and the other II-point lies at infinity. In this case, therefore, the envelope becomes a parabola. We thus obtain the well-known theorem that the envelope of the Simson lines of a point is a parabola escribed to the triangle and having the point as focus.

It is obvious that the foregoing proof establishes the more general theorem that the envelope of a coaxaloid system of circles, without restriction as to connection with the triangle, is a conic having the II-points of the system as foci, and the minimum circle of the system as auxiliary circle.

12. The envelope of a coaxaloid system of circles connected with the triangle, touches the sides of the triangle at the points where these are met by the lines joining the S-point to the opposite vertices.

Demonstration. Let ABC (Fig. 10) be the triangle, and S any point in its plane. Let AS, BS, CS meet BC, CA, AB in D, E, F respectively. Then by Art 3 (ii) one of the circles of the coaxaloid system connected with the triangle and having S as S-point, touches BC at D. This is sufficient to prove that the envelope of the system also touches BC at D. Similarly it might be shown that it touches CA and AB at E and F respectively.

We are now in a position to find a geometrical solution of the following problem: Given the S-point of a coaxaloid system of circles connected with the triangle ABC, to find the II-points, and thus determine the system generally.

Solution. Let X, Y, Z (Fig. 10) be the mid-points of the connectors EF, FD, DE respectively. Then since ABC is circumscribed to the envelope of the system and DEF is inscribed in it, the connectors AX, BY, CZ meet in the centre of the envelope. Let this point be denoted by K. It may be noted that S and K coincide when S is the centroid of ABC. If now two points J and J′ be taken on AK, on opposite sides of K, such that $KJ^2 = KJ'^2 = KX \cdot KA$, JJ′ is obviously the diameter of the envelope conjugate to the chord EF. Being given a chord and the conjugate diameter, we can find the foci of the envelope by any one of a number of well-known

constructions, *i.e.*, we can find the Π-points of the system. Then by drawing perpendiculars Πl, Πm, Πn, Π'L, Π'M, Π'N from these points to the sides of ABC as in Fig. 8, we determine one circle of the system, and by drawing parallels (or antiparallels) to the connectors Mn, Nl, Lm so as to intersect two and two on CS, AS, BS we determine the other circles of the system.

The following particular case of the problem just solved may be noticed: Given a point S in the plane of a triangle, to draw through S three lines terminated by the sides of the triangle, two and two, such that their extremities shall lie on a circle.

Combining the theorem of the present article with the proposition that every six-point circle connected with the triangle belongs, in general, to four coaxaloid systems connected with the triangle, we obtain the following interesting result. Every six-point circle connected with the triangle has, in general, double contact (real or imaginary) with four conics which touch the sides of the triangle at the points, taken three and three, where the connectors of the vertices with the four S-points meet the opposite sides.

13. Every coaxaloid system of circles having real Π-points, is an S-point system of an infinite number of triangles, viz., of all the triangles circumscribed to the envelope of the system.

Demonstration. Let Π and Π' be the Π-points of a coaxaloid system, and let any triangle be circumscribed to its envelope. The minimum circle of the system is the auxiliary circle of the envelope, and therefore, by a well-known property of tangents to a conic, is the pedal circle of Π and Π' with respect to the triangle. Again, by another well-known property of tangents to a conic, Π and Π' are isogonal conjugates with respect to the triangle, and hence are the Π-points of an S-point system connected with the triangle. Of this system the pedal circle of Π and Π' with respect to the triangle is one of the circles. Hence the ratio of both systems—the coaxaloid and the S-point system—is the diameter of the pedal circle of Π and Π' over the distance ΠΠ'. Therefore, since both systems have the same Π-points and the same ratio, they are identical, which proves the proposition.

The theorem is important as affording a starting-point for the discussion of coaxaloid systems of circles connected with the quadrilateral and general polygon.

14. The tangent to the minimum circle of a coaxaloid system (with real Π-points) at either extremity of the diameter through Π and Π′ meets every other circle of the system in two points which connect with the centre of the latter through Π and Π′.

Demonstration. Let K (Fig. 9) be the mid-point of ΠΠ′, and, therefore, the centre of the minimum circle, and K′ the centre of any other circle of the system. Let the tangent to the minimum circle at T, one of the extremities of the diameter through Π and Π′, meet the connectors ΠK′, Π′K′ in U and U′ respectively. Then by *Euc.* VI., 2, K′U/K′Π = KT/KΠ. Therefore, by Art. 10, U lies on the circle K′. Similarly K′U′/K′Π′ = KT/KΠ′. Therefore U′ also lies on the circle K′.

15. The circles of the coaxal system from which a coaxaloid system is derived, are the circles of similitude of the circles of the coaxaloid system taken in pairs.

Demonstration. Let K′, K″ (Fig. 9) be the centres of two circles of the coaxaloid system, E their external and I their internal centre of similitude. Then EK′/EK″ = K′I/IK″

= radius of circle K′/radius of circle K″

= ΠK′/ΠK″.

Therefore EI subtends a right angle at Π, which proves the proposition.

In the proof Π and Π′ are real points, but the same result holds good when they are imaginary.

16. The locus of the extremities of a system of parallel diamaters of a coaxaloid system of circles is a hyperbola.

Demonstration. Through K, the mid-point of ΠΠ′, (Fig. 9) draw any line KV, and from K′, the centre of any circle of the coaxaloid system, draw K′W a radius, parallel to KV. Then if x and y are the coordinates of W with respect to the axes KV, KK′, we have

$$x^2 = K'W^2 = K'\Pi^2 . KT^2/K\Pi^2$$
$$= (K'K^2 + K\Pi^2)KT^2/K\Pi^2$$
$$= (y^2 + K\Pi^2)KT^2/K\Pi^2.$$

Hence the equation of the locus of W is $x^2/KT^2 - y^2/K\Pi^2 = 1$, which proves the proposition.

By varying the direction of the parallel diameters we obtain a system of hyperbolas associated with the coaxaloid system. These are obviously concentric, with K as centre, and have a common diameter equal to ΠΠ′ coinciding with the line of centres of the coaxaloid system. Obviously also their diameters conjugate to this common diameter are diameters of the minimum circle of the coaxaloid system; hence the tangents to the hyperbolas at the points where the latter are cut by the minimum circle are perpendicular to ΠΠ′. When the coaxaloid system is coaxal, KT = KΠ, and hence the associated hyperbolas are all equilateral. When the diameters of the circles are drawn parallel to ΠΠ′, the hyperbola on which their extremities lie has TT′, the diameter through Π and Π′ of the minimum circle, as transverse axis, and the common diameter of all the hyperbolas, as conjugate axis. In this case the minimum circle of the coaxaloid system is the auxiliary circle of the hyperbola.

The theorem has been proved for the case when Π and Π′ are real, but might easily be extended to the case when they are imaginary.* In the former case the extremities of the parallel diameters of the circles lie on different branches of the hyperbolas, and the common diameter of the latter is ideal; in the latter case the extremities of the parallel diameters of the circles lie on the same branches of the hyperbolas and the common diameter of the latter is real.

The converse theorem may be stated: Circles described on parallel chords of a hyperbola as diameters form a coaxaloid system, with real Π-points if the extremities of the chords lie on different branches of the curve, with imaginary Π-points if they lie on the same branch. In the particular case when the hyperbola is a line-pair, the coaxaloid circles reduce to circles having a common centre of similitude, and the other associated hyperbolas are also line-pairs.

17. The hyperbolas associated with a coaxaloid system have the same envelope as the system.

Demonstration. If KV, KK′ (Fig. 9) be taken as axes, the equation of the hyperbola obtained by drawing diameters of the system parallel to KV is $x^2/a^2 - y^2/c^2 = 1$, where $a = KT$, and $c = K\Pi$.

* See note at end of paper.

Transforming to the axes KT, KK′, we have

$$x^2\sec^2\phi/a^2 - (y - x\tan\phi)^2/c^2 = 1,$$

where ϕ denotes the angle VKT′. Rewriting we have

$$x^2(a^2 - c^2)\tan^2\phi - 2a^2xy\tan\phi - c^2x^2 + a^2y^2 + a^2c^2 = 0.$$

Hence the equation of the envelope of the hyperbolas is

$$(a^2 - c^2)(a^2y^2 - c^2x^2 + a^2c^2) = a^4y^2,$$

$$i.e. \quad x^2/a^2 + y^2/(a^2 - c^2) = 1,$$

which represents the conic having Π, Π′ as foci, and TT′ as transverse axis.

II.

Particular Cases of Coaxaloid Systems of Circles connected with the Triangle.

Some of the more obvious particular cases of coaxaloid systems connected with the triangle ABC will now be noticed.

18. *The Tucker Circles.* When the angles α, β, γ (Art. 4) equal B, C, A (or C, A, B) respectively, the trilinear coordinates of S are

$$\sin A, \quad \sin B, \quad \sin C;$$

i.e., S is the symmedian point.

By Art. 5, the trilinear coordinates of the Π-points are

$$c/b, \quad a/c, \quad b/a \quad \text{and} \quad b/c, \quad c/a, \quad a/b$$

i.e., the Π-points are the Brocard points.

This is the well-known case of the Tucker circles.

The ratio of this system may be calculated in terms of the angles of ABC as follows: If Ω, Ω′ are the Brocard points, and ΩL, Ω′L′ be drawn perpendicular to BC, we have, by Art. 11 (vii)

$$r^2 = 1 + 4\Omega L \,.\, \Omega'L'/\Omega\Omega'^2.$$

Hence, using the well-known results,

$$\Omega L \,.\, \Omega'L' = 4R^2\sin^4\omega,$$

$$\Omega\Omega'^2 = 4R^2\sin^2\omega(1 - 4\sin^2\omega),$$

and

$$\operatorname{cosec}^2\omega = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C,$$

where R is the radius of the circumcircle, and ω the Brocard angle, we have

$$r^2 = (1 - 3\sin^2\omega)/(1 - 4\sin^2\omega)$$

$$= \Sigma\cot^2 A/(\Sigma\cot^2 A - 1).$$

19. *A System derived from the Nine-Point Circle.*

If A′, B′, C′ are the mid-points of BC, CA, AB, and D, E, F the feet of the altitudes on these sides respectively, then the connectors FB′, DC′, EA′ are the directive chords of one of the four systems derivable from the nine-point circle.

Since $\widehat{AFB'} = A$, $\widehat{BDC'} = B$, and $\widehat{CEA'} = C$,

the angles of the system are A, B, C (or $\pi - 2A$, $\pi - 2B$, $\pi - 2C$). Hence, by Art. 4, the trilinear coordinates of S are

$$\sec A/a^2,\ \sec B/b^2,\ \sec C/c^2;$$

i.e., S is the isotomic conjugate of the circumcentre. Thus the other circles of the system are obtained by drawing transversals M*n*, N*l*, L*m*, terminated by the sides as in Fig. 6, so as to intersect two and two on CS, AS, BS, where S is the isotomic conjugate of the circumcentre, and so as to make with the sides of ABC three isosceles triangles whose vertices are M, N, L (or *m*, *n*, *l*) respectively.

By Art. 5 the trilinear coordinates of the H-points are

$$\cos A,\ \cos B,\ \cos C \text{ and } \sec A,\ \sec B,\ \sec C;$$

i.e., the H-points are the circumcentre, O, and the orthocentre, H. Thus the line of centres of the system is the perpendicular bisector of OH, and the nine-point circle, being the pedal circle of O and H, is the minimum circle of the system.

By Art. 10 (viii), and the well-known relations,

$$OA' = R\cos A,\quad HD = 2R\cos B\cos C,$$

and
$$OH^2 = R^2(1 - 8\cos A\cos B\cos C),$$

we have
$$r^2 = 1/(1 - 8\cos A\cos B\cos C).$$

It may be noticed that the following theorem is involved in the foregoing: The connector of A with the isotomic conjugate of O contains the points of intersection of the pairs DC′, EA′ and FA′, DB′; with similar theorems for the vertices B and C.

20. *Other Systems derived from the Nine-Point Circle.*

Take B′F, C′A′, DE (notation of preceding article) as directive chords. Since $\widehat{AFB'} = A$, $\widehat{BA'C'} = C$, and $\widehat{CED} = B$, the angles of the resulting system are A, C, B (or $\pi - 2A$, A, A). Hence by Art. 4 the trilinear coordinates of S are

$$\operatorname{cosec} 2A,\ \operatorname{cosec} C,\ \operatorname{cosec} B,$$

by which it at once appears that AS is the symmedian through A, and the following construction, independent of the nine-point circle, is suggested. On CA, AB describe isosceles triangles VCA, WAB having their base angles equal to A. Then BV, CW and the symmedian through A cointersect in S.

By Art. 5 the trilinear coordinates of Π are

$$\sin A \cos ec 2A,\ \sin C \cos ec A,\ \sin B \cos ec A,$$

from which it is evident that Π lies on the median through A, and that, consequently, Π′ lies on the corresponding symmedian, and, therefore on AS. Since $B\widehat{\Pi}C = \pi - A$ (Art. 5), and $B\widehat{\Pi'}D = 2A$, the median through A meets the circumcircle of BHC in Π, and the symmedian through A meets the circumcircle of BOC in Π′. This suggests a construction for finding Π and Π′. The line of centres, which, it may be noted, passes through the nine-point centre, may be found by drawing the perpendicular bisector of the distance between Π and Π′, or it may be found by the following construction. Let K be the symmedian point of ABC, and at L where AK meets BC draw a perpendicular to BC meeting OK in T. Then the connector of T with the nine-point centre is the line of centres. This may be proved as follows. Through L draw LM, L*m* to meet CA, and LN, L*n* to meet AB, so that L*m* shall be parallel to AB and LM antiparallel to it with respect to C, and that LN shall be parallel to CA, and L*n* antiparallel to it with respect to B. Then by Art. 3 (ii) since $B\widehat{L}N = C$, and $C\widehat{M}L = B$, and L*n*, L*m* are the antiparallels of LN, LM respectively, the points L, M, *m*, N, *n* lie on a circle of the system which touches BC at L, and whose centre, therefore lies on LT. But since AL is a symmedian, and $B\widehat{L}N = C$, and $C\widehat{m}L = A$, and L*n*, LM are the antiparallels of LN and L*m* respectively, this circle is also a Tucker circle, and, therefore, by a well-known theorem relating to Tucker circles, its centre lies on OK. Therefore T is the centre of the circle, and since the nine-point circle is another circle of the system, the statement is proved.

Two other analogous systems may be derived from the nine-point circle by taking

(i) EF, C′D, A′B′ and (ii) B′C′, FD, A′E

as directive chords.

The following theorem is easily deducible from the foregoing:

The point of intersection of C'A' and DE, and also the point of intersection of the antiparallel pair DF and A'B' lie on the symmedian AK; with similar statements for BK and CK.

21. *The Three Rectangular Systems.*

The circumcircles of all the rectangles inscribed in ABC so as to have one of their sides lying on a side of the triangle, form a coaxaloid system connected with the triangle.

Demonstration. Let lLmN (Fig. 11) be any rectangle inscribed in ABC, having lL lying on BC, m on CA, and N on AB; and let the circumcircle of the rectangle meet CA again in M, and AB again in n. Then the connector Mn is antiparallel to mN, and therefore to BC, with respect to A. Therefore the chords Mn, Nl, Lm are constant in direction, which proves the proposition.

The system thus determined may be called the A rectangular system. Two other analogous systems are obtained by taking the rectangles so as to have a side coinciding (i) with CA and (ii) with AB. These may be called the B and C rectangular systems respectively.

Since, as is well-known, the six points in which the Cosine circle meets the sides of the triangle can be joined so as to form three rectangles having each a side coincident with a side of the triangle, the Cosine circle belongs to each of the rectangular systems. Hence the three lines of centres of the system cointersect in the centre of the Cosine circle, *i.e.*, in the symmedian point.

The angles of the A system are obviously

$$\mathrm{C},\ \frac{\pi}{2},\ \frac{\pi}{2}-\mathrm{C}\ \left(\text{or } \mathrm{B},\ \frac{\pi}{2}-\mathrm{B},\ \frac{\pi}{2}\right).$$

Hence, by Art. 4, the trilinear coordinates of the S-point of this system are

$$\cos ec\mathrm{B}\cos ec\mathrm{C},\ \sec\mathrm{B},\ \sec\mathrm{C},$$

by which it is evident that S lies on the altitude AD, and the following construction is suggested: From D, the foot of the altitude AD, draw perpendiculars to CA and AB meeting the parallel to BC through A in V and W respectively; then AD, BV, CW cointersect in S.

The line of centres can be readily determined without first

finding the Π-points. The side BC and the altitude AD are degenerate forms of the rectangles which give rise to the A system, and, therefore, the line of centres of the system is the line joining the mid-points of BC and AD.

To solve the problem of finding the Π-points of the A system, let $lLmN$ (Fig. 11) be the rectangle inscribed in the minimum circle of the system. From l draw a perpendicular to CA meeting CA in M and Lm in Π′; and from L draw a perpendicular to AB meeting AB in n and Nl in Π. Then it is obvious that M and n are points on the circumcircle of $lLmN$, and that Π and Π′, being the points of concurrence of perpendiculars to the sides at the points where these are cut by the minimum circle, are the Π-points. Now $\Pi l/lL = \cot B$, and $\Pi'L/lL = \cot C$. But since Π′N and Ln are perpendicular to AB, $\Pi'L = \Pi N$.

Therefore $$Nl/lL = \cot B + \cot C = BC/AD.$$

This result determines the species of the rectangle $lLmN$, which can, consequently, be easily constructed. The Π-points are then obtained by drawing lM, Ln perpendicular to CA, AB respectively as above.

To determine the ratio of the A system we have

$$\Pi l \,.\, \Pi'L = lL^2 \cot B \cot C,$$

and
$$\Pi\Pi'^2 = lL^2 + (Nl - 2\Pi'L)^2$$
$$= lL^2\{1 + (\cot B - \cot C)^2\}.$$

Hence, by Art. 11 (vii) we obtain

$$r^2 = \frac{1 + (\cot B + \cot C)^2}{1 + (\cot B - \cot C)^2}$$

The following is given as an example of the numerous theorems dealing with collinearity and concurrency, which arise in connection with these systems. If $lLMn$, $mMNl$, $nNLm$ are the three inscribed rectangles of the Cosine circle, lL, mM, nN coinciding with BC, CA, AB respectively, and AD, BE, CF the altitudes, the following are triads of concurrent lines,

$$AD,\ NL,\ lm;\quad BE,\ LM,\ mn;\quad CF,\ MN,\ nl.$$

This follows at once from the fact that pairs of directive chords of the rectangular systems intersect on the altitudes.

22. *The Four Concentric Systems.*

When to α, β, γ are assigned the values

$$\tfrac{1}{2}(B+C),\ \tfrac{1}{2}(C+A),\ \tfrac{1}{2}(A+B)$$

respectively, *i.e.*, when the directive chords Mn, Nl, Lm (Fig. 12) make with the sides of ABC three isosceles triangles whose vertical angles are the interior angles A, B, C respectively, the trilinear coordinates of S are found by Art. 5 (i) to be

$$\sec^2\frac{A}{2},\quad \sec^2\frac{B}{2},\quad \sec^2\frac{C}{2};$$

i.e., S is the point of concurrence of the lines joining the vertices of ABC to the points where the incircle touches the opposite sides (Gergonne point).

Since the directive chords in this case are isoclinal to the sides of the triangle taken in pairs, it follows from the last paragraph of Article 8 that the circles of the system are concentric.

The incircle evidently belongs to this system, for if X, Y, Z (Fig. 12) are the points where this circle touches BC, CA, AB respectively, the chords YZ, ZX, XY are obviously parallel to the directive chords of the system. It follows that the incentre is the common centre of the system.

By Art. 5, or otherwise, the Π-points of the system are found to coincide in the incentre. Hence the incircle besides being the minimum circle is the (ideal) envelope of the system.

It may be noticed also that α and $\pi-(A+\alpha)$ are in this case equal, and so for the other pairs, and that, consequently, the two variable inscribed triangles of constant species which by their rotation (round the incentre) determine the system, are similar, their angles opposite the vertices A, B, C being equal to $\frac{1}{2}(B+C)$, $\frac{1}{2}(C+A)$, $\frac{1}{2}(A+B)$ respectively.

By Art. 10 (vii), $\qquad r=\infty.$

The following particulars may be noted with respect to the figure of the present article (Fig. 12)

(i) mN, nL, lM are both parallel and antiparallel (with respect to A, B, C) to Mn, Nl, Lm respectively.

(ii) Ll = Mm = Nn, and X, Y, Z are their mid-points.

(iii) If P, Q, R are the points on AS, BS, CS respectively,

where Mn, Nl, Lm intersect, and p, q, r the points on AS, BS, CS where mN, nL, lM intersect,

$$\widehat{RPQ} = \widehat{rpq} = \tfrac{1}{2}(B + C),$$
$$\widehat{PQR} = \widehat{pqr} = \tfrac{1}{2}(C + A),$$
$$\widehat{QRP} = \widehat{qrp} = \tfrac{1}{2}(A + B).$$

Hence Ll, Mm, Nn are antiparallel to QR, RP, PQ with respect to the angles RPQ, PQR, QRP respectively, and to qr, rp, pq with respect to the angles rpq, pqr, qrp respectively. They are also antiparallel to YZ, ZX, XY with respect to the angles ZXY, XYZ, YZX respectively.

(iv) S is the symmedian point of the triangles PQR, pqr, XYZ.

(v) The circle $LlMmNn$ is a Tucker circle of the triangles PQR, pqr, XYZ.

(vi) Hence, if I be the incentre of ABC, J the circumcentre of PQR and J′ the circumcentre of pqr, then I, S, J, J′ are collinear, and I is the mid-point of JJ′.

By taking S as the point of concurrence of the lines joining A, B, C to the points where the excircle opposite to A touches the opposite sides, and drawing Mn, Nl, Lm so as to make with the sides of ABC three isosceles triangles whose vertical angles are the interior angle at A and the exterior angles at B and C, we obtain an analogous system concentric with the excircle and including it. An analogous system exists in connection with each of the other excircles.

23. *Systems connected with the Isogonic Centres.*

When $\alpha = \beta = \gamma = 60°$ (or $\alpha = 120° - A$, $\beta = 120° - B$, $\gamma = 120° - C$), the trilinear coordinates of S are found by Art. 4 to be

$$\operatorname{cosec}(60° + A),\ \operatorname{cosec}(60° + B),\ \operatorname{cosec}(60° + C).$$

Hence S is the point of concurrence of the lines joining the vertices of ABC to the vertices of equilateral triangles described externally on the opposite sides, *i.e.*, it is one of the isogonic centres of ABC.

Using Art. 5 we find that, in the case of this system, Π coincides with the isogonic centre. Hence the line of centres of the system is the perpendicular bisector of the distance between the isogonic centre and its isogonal conjugate.

It follows from Art. 7 that the circles of this system are the circumcircles of a variable equilateral triangle inscribed in ABC.

When two of the angles α, β, γ are taken each equal to 120°, and the remaining angle to $-60°$, an analogous system is obtained for which the trilinear coordinates of both S and Π are found to be

$$\cos ec(60° - A), \cos ec(60° - B), \cos ec(60° - C).$$

Thus S and Π coincide in the point of concurrence of the lines joining the vertices of ABC to the vertices of equilateral triangles described internally on the opposite sides, *i.e.*, in the other isogonic centre of ABC. In this case the directive chords are inclined at angles of 60° to the corresponding chords in the previous case.

24. *Coaxal Systems.*

It has been remarked in Art. 10 (ii) that a coaxaloid system connected with the triangle reduces to a coaxal system, when the Π-points lie on the perimeter of the triangle. In illustration the cases which arise in connection with the circle described on a side as diameter may be referred to.

Let AD, BE, CF be the three altitudes of ABC, and consider the circle described on BC as diameter. The four systems to which this circle belongs may be obtained by taking the following sets of directive chords,

(i) CB, FC, BE, which give $\alpha = B, \beta = \frac{\pi}{2} - B, \gamma = \frac{\pi}{2}$,

(ii) CF, BC, BE, which give $\alpha = \frac{\pi}{2}, \beta = 0, \gamma = \frac{\pi}{2}$,

(iii) EB, FC, CB, which give $\alpha = \frac{\pi}{2} - A, \beta = \frac{\pi}{2} - B, \gamma = \pi - C$,

(iv) EF, BC, CB which give $\alpha = C, \beta = 0, \gamma = \pi - C$.

In the first case we have the A-rectangular system (Art. 21), and in the second, third, and fourth cases we have the coaxal systems which have BE, CF and BC, respectively, as common chords.

III.

Additional Theorems.

25. The envelope of the polars of a point with respect to the circles of a coaxaloid system is a parabola.

Demonstration. The equation of any circle of the system referred to ΠΠ′ and KK′ as axes (Fig. 9) is

$$x^2 + (y - c\tan\phi)^2 = a^2\sec^2\phi,$$

where $KT' = a$, $K\Pi' = c$, and ϕ denotes the angle subtended at Π′ by the distance of the centre of the circle from K.

The equation of the polar of a point x', y' with respect to the circle is

$$(c^2 - a^2)\tan^2\phi - c(y + y')\tan\phi + xx' + yy' - a^2 = 0.$$

The equation of the envelope of this, when ϕ varies, is

$$4(c^2 - a^2)(xx' + yy' - a^2) = c^2(y + y')^2,$$

which represents a parabola.

When the system is coaxal, $c = a$, and the parabola, as is well known, reduces to a point.

26. The locus of the poles of a line with respect to the circles of a coaxaloid system is, in general, a hyperbola, but when the line is parallel to the line of centres it is a parabola.

Demonstration. Let the line meet the line of centres at O (Fig. 9) and make with it an angle α. Let K′M be the perpendicular to the line from K′, the centre of a circle of the system, and let P be the pole of the line with respect to the circle K′.

Let $K\Pi = c$, $OK = d$, and $r =$ the ratio of the system.

OK is taken as the axis of x, and the line through O parallel to the fixed direction K′M as the axis of y.

Then $K'P \cdot K'M = (\text{radius of circle } K')^2 = r^2 . K'\Pi^2 = r^2(c^2 + K'K^2)$.
$= r^2\left(c^2 + \overline{K'O - d}^2\right)$.

And $K'M = K'O \cdot \sin\alpha$.

Hence we obtain as the equation of the locus of P,

$$yx\sin\alpha = r^2\left(c^2 + \overline{x - d}^2\right),$$

$$i.e., \quad r^2x^2 - xy\sin\alpha - 2r^2dx + r^2(c^2 + d^2) = 0,$$

which represents a hyperbola, except when $\sin\alpha = 0$, when it represents a parabola.

This theorem holds also when the circles are coaxal.

27. The envelope of the radical axes of any circle and the circles of a coaxaloid system is a parabola.

Demonstration. As in Art. 25 the equation of any circle of the coaxaloid system referred to IIII′ and KK′ as axes (Fig. 9) is

$$x^2 + (y - c\tan\phi)^2 = a^2\sec^2\phi \quad - \quad - \quad - \quad (1).$$

Let the equation of a circle not belonging to the system be

$$(x-p)^2 + (y-q)^2 = r^2 \quad - \quad - \quad - \quad (2).$$

Then the equation of the radical axis of (1) and (2) is

$$(c^2 - a^2)\tan^2\phi - 2cy\tan\phi + 2px + 2qy + r^2 - p^2 - q^2 - a^2 = 0.$$

Therefore the equation of the envelope is

$$c^2y^2 = (c^2 - a^2)(2px + 2qy + r^2 - p^2 - q^2 - a^2),$$

which represents a parabola.

When the system is coaxal, $c = a$, and the parabola, as is well known, reduces to a point.

28. Suppose that CA in Fig. 8 is made to move, in contact with the envelope of the system, until it becomes parallel to BC.

Then, in this case, the angles L′Π′M′ and $l'\Pi m'$ which are supplementary to the angle C, become each equal to two right angles, so that L′M′, $l'm'$ pass through Π′ and Π respectively.

Again, since in the general case, Π′L′, Πl' are equally inclined to BC, L′M′, $l'm'$ are equally inclined to BC and CA, and therefore are equal. Hence the segments, known to be of constant form, cut off from the variable circle L′l'M′m' by L′M′ and $l'm'$ are, in this case, equal for each position of the circle.

Also when the circle L′l'M′m' is the auxiliary circle of the envelope, L′M′, $l'm'$ are perpendicular to BC and CA.

Therefore L′m' = l'M′ = diameter of the auxiliary circle

= focal axis of the envelope.

But since BC, CA are parallel, L′m', l'M′, already known to be fixed in direction, are also constant in length for all positions of the circle L′l'M′m', and are therefore each equal to the focal axis of the envelope. From this it follows that they are equally inclined to BC and CA. By making L′ and l' coincide at D, the point where BC touches the envelope, we see that the fixed directions of L′m', l'M′ are parallel to IID, Π′D respectively.

The foregoing may be summed up in the following theorem, of which that of Art. 15 is a particular case.

If two fixed parallel tangents to the envelope of a coaxaloid system whose Π-points are Π and Π′, meet a variable circle of the system in the points L, *l* and M, *m*, then

(i) Two of the connectors of these four points, say LM and *lm*, pass through Π′ and Π respectively, and LM, *lm* are equally inclined to L*l* and M*m*, and cut off two pairs of equal segments of constant form from the circle; and

(ii) L*m*, *l*M are constant in length being each equal to the focal axis of the envelope, and are fixed in direction being parallel to ΠD, Π′E and Π′D, ΠE respectively, where D and E are the points of contact of L*l* and M*m* with the envelope.

It might be easily proved also that

$$LD/Dl = l\Pi/\Pi m = ME/Em = M\Pi'/\Pi'L = \Pi'D/D\Pi = \Pi E/E\Pi'.$$

The following theorem is deducible from the foregoing as a particular case.

If a circle have double contact with a conic whose foci are Π and Π′, and lines be drawn from L, one of the points of contact, through Π and Π′ meeting the circle in M and *m* respectively, then M*m* is the tangent to the conic parallel to the tangent at L, and LM = L*m* = the focal axis of the conic.

Another interesting particular case might be obtained by supposing that L*l* and M*m* coincide on an asymptote to the envelope.

29. If LMN (Fig. 13) be a variable triangle of constant species inscribed in the triangle ABC, not only the circumcircle of LMN, but any circle invariably connected with it (such as the incircle, Taylor circle, Brocard circle, etc.) forms, in its various positions, a coaxaloid system.

Demonstration. Let Π be the centre of similitude of the triangle LMN in its various positions. Draw through Π three lines making constant angles with MN, NL, LM respectively, and meeting the circle in *l*, *m*, *n* and the sides BC, CA, AB in *l*′, *m*′, *n*′ respectively.

Then the ratios ΠL/Π*l*, etc., are constant, and also the ratios ΠL/Π*l*′, etc. Therefore the ratios Π*l*/Π*l*′, etc., are constant. Hence *lmn* is a triangle of constant species, and *l*, *m*, *n* move on straight lines parallel to BC, CA, AB respectively. That is the circle is the circumcircle of a variable triangle of constant species inscribed in a fixed triangle, and, therefore, by Art. 7, forms a coaxaloid system.

If the circle passes through Π the system is coaxal; if LMN is similar to the orthocentric triangle of ABC, and the circle the incircle of LMN, the system is concentric.

30. If a system of directly similar triangles have the same centre of similitude when considered in pairs, the circle passing through the points where the sides of a variable triangle of the system meet the homologous sides of a fixed triangle of the system, forms, in its various positions, a coaxaloid system having the centre of similitude as one of its Π-points.

Demonstration. In the figure (Fig. 14) Π is the common centre of similitude, ABC the fixed triangle, A'B'C' the variable triangle similar to it, L, M, N the points where BC, CA, AB are intersected by the homologous sides of A'B'C' respectively, and l, m, n the feet of the perpendiculars from Π on BC, CA, AB.

Then, since ΠAB, ΠA'B' are equal angles, Π, A, A', N are concyclic, and since MAN, MA'N are equal angles, M, A, A', N are concyclic. Therefore Π, M, A, N are concyclic. Therefore MΠN, $m\Pi n$ are equal angles. Similarly it may be shown that NΠL, $n\Pi l$ are equal angles, and also LΠM, $l\Pi m$. Therefore LΠl, MΠm, NΠn are equal angles. Hence it might easily be shown that the triangle LMN is similar to the triangle lmn. Therefore, by Art. 8, its circumcircle forms, in its various positions, a coaxaloid system having Π as one of the Π-points.

31. If a number of coaxaloid systems have the same ratio r, and have besides a common Π-point, Π, while the other Π-points lie on any locus, say a straight line, they form a web of circles possessing the following obvious properties.

(i) If a transversal be drawn in any direction across the web, the circles whose centres lie on it form a coaxaloid system one of whose Π-points is Π, and whose ratio is r; for if the transversal pass through K_1, K_2, K_3 ... centres of circles of the web, and ρK denote the radius of the circle whose centre is K, the ratios $\rho K_1/K_1\Pi$, $\rho K_2/K_2\Pi$, $\rho K_3/K_3\Pi$ ... are each equal to r.

(ii) When the transversal is drawn through Π, the circles whose centres lie on it have Π as a common centre of similitude; for if M_1, M_2, M_3 ... be centres lying on the transversal,

$$\rho M_1/M_1\Pi = \rho M_2/M_2\Pi = \ldots = r.$$

(iii) The centres of equal circles of the web lie on a circle whose centre is Π; for if C_1, C_2, C_3 ... be centres of equal circles,

$$\rho C_1/C_1\Pi = \rho C_2/C_2\Pi = \rho C_3/C_3\Pi = \ldots = r,$$

and therefore since $\rho C_1 = \rho C_2 = \rho C_3 =$ etc.,

$$C_1\Pi = C_2\Pi = C_3\Pi = \text{etc.}$$

A web such as has been described may be called a coaxaloid web of the first kind. It may be generated in a great number of different ways, of which the two following may be mentioned.

(1) If a variable triangle of constant species have a fixed point Π invariably connected with it, a circle invariably connected with it, forms, in its different positions, a coaxaloid web of the first kind; for, obviously, the radius of the circle bears a constant ratio to the distance of its centre from Π.

(2) If in the system of triangles referred to in the last article, different triangles be taken in succession as the fixed triangle, the coaxaloid systems so obtained form a web of the first kind.

Demonstration. The centre of similitude, Π, of the triangles is invariably connected with them. Therefore its pedal triangle with respect to each of them is of constant species. Therefore the triangle determined by the intersections of homologous sides of every pair of them is also of constant species, and has Π invariably connected with it. Therefore by (1) above, its circumcircle forms a web of the first kind.

32. If a variable line terminated at Π and Π' move parallel to itself so that Π and Π' always lie on two fixed straight lines intersecting in S, a coaxaloid system of circles having Π and Π' as Π-points and a constant ratio r, generates, as $\Pi\Pi'$ varies, a web of circles which may be called a coaxaloid web of the second kind, and which possesses the following properties.

(i) Those circles of the web whose centres lie on any transversal through S, have S as a common centre of similitude.

(ii) Generally the circles whose centres lie on any transversal are coaxaloid.

(iii) The coaxaloid systems determined by parallel transversals have the same ratio, and have their Π-points lying on two straight lines through S.

(iv) Systems of equal circles belonging to the web have their centres lying on a series of similar and similarly placed ellipses with S as centre.

Demonstration. Let $\Pi_1\Pi'_1$ and $\Pi_2\Pi'_2$ (Fig. 15) be two positions of $\Pi\Pi'$, and let O_1, O_2 be their respective mid-points. Let the perpendiculars at O_1, O_2 to $\Pi_1\Pi'_1$ and $\Pi_2\Pi'_2$ (the lines of centres of the coaxaloid systems whose II-points are Π_1, Π'_1 and Π_2, Π'_2 respectively), meet a transversal through S in C_1, C_2, and a parallel transversal in K_1, K_2 respectively. Through S draw a parallel to O_1K_1, meeting K_1K_2 in K. It will be noticed that SK is the line of centres of that one of the original coaxaloid systems whose II-points coincide in S. Let ρC denote the radius of the circle (belonging to the web) whose centre is C.

(i) $C_1S/C_2S = O_1S/O_2S = \Pi_1S/\Pi_2S$.

Therefore $C_1\Pi_1$ and $C_2\Pi_2$ are parallel.

Therefore $C_1\Pi_1/C_1S = C_2\Pi_2/C_2S$.

But $\rho C_1/C_1\Pi_1 = \rho C_2/C_2\Pi_2 = r$.

Therefore $\rho C_1/C_1S = \rho C_2/C_2S$.

Therefore S is the centre of similitude of the circles C_1 and C_2. In the same way it may be proved to be the centre of similitude of every other pair of circles whose centres lie on the same transversal.

(ii) Let SK, Π_1K_1 meet at Q, and let the circumcircles of $QS\Pi_1$, QKK_1 meet again in P. Then since P, K, K_1, Q are concyclic, PKS and $PK_1\Pi_1$ are equal angles and since P, S, Π_1, Q are concyclic, PSK and $PI\Pi_1K_1$ are equal angles. Therefore PKS and $PK_1\Pi_1$ are similar triangles. - - - - - (1).
Hence PKK_1 and $PS\Pi_1$ are similar triangles, and, therefore, since $KK_1/KK_2 = S\Pi_1/S\Pi_2$, PKK_2 and $PS\Pi_2$ are similar triangles. Hence PKS and $PK_2\Pi_2$ are similar triangles. - - - (2).

Therefore, by (1) and (2) $KS/KP = K_1\Pi_1/K_1P = K_2\Pi_2/K_2P$.

But $\rho K/KS = \rho K_1/K_1\Pi_1 = \rho K_2/K_2\Pi_2 = r$.

Therefore $\rho K/KP = \rho K_1/K_1P = \rho K_2/K_2P$.

Therefore the circles (of the web) whose centres lie on the tranversal KK_1 are coaxaloid, with P as a II-point.

(iii) $\widehat{KPK_1} = \widehat{KQK_1} = \widehat{C_1K_1\Pi_1}$,

and $PK/PK_1 = KS/K_1\Pi_1 = K_1C_1/K_1\Pi_1$.

Therefore PKK$_1$ and K$_1$C$_1$Π$_1$ are similar triangles.

Therefore $PK/K_1C_1 = KK_1/C_1\Pi_1$,

i.e. $PK/KS = SC_1/C_1\Pi_1$.

But $PK/PS = KK_1/S\Pi_1 = SC_1/S\Pi_1$.

Therefore PKS and SC$_1$Π$_1$ are similar triangles.

Let r' denote the ratio of the system of circles C$_1$, C$_2$..., and r'' the ratio of the system K, K$_1$, K$_2$....

Then $\rho C_1/C_1\Pi_1 = r$, and $\rho C_1/C_1S = r'$.

Therefore $r' = r \cdot C_1\Pi_1/C_1S$.

And $\rho K/KS = r$, and $\rho K/KP = r''$.

Therefore $r'' = r \cdot KS/KP$.

But $KS/KP = C_1\Pi_1/C_1S$.

Therefore $r'' = r'$.

Hence systems whose centres lie on parallel transversals have the same ratio.

Again, since PKS and SC$_1$Π$_1$ are similar triangles, the angle KSP is equal to the angle C$_1$Π$_1$S which is constant so long as SC$_1$ retains its direction. Therefore the locus of P, as KK$_1$ moves parallel to SC$_1$ is a straight line through S. Hence it may be readily proved that the locus of the image of P with respect to KK$_1$, *i.e.*, the other Π-point, is also a straight line through S. Thus statement (iii) is completely proved.

(iv) Let K$_1$ be the centre of a variable circle of the web, having a constant radius ρ; and regard the circle as belonging in its different positions to the variable coaxaloid system whose line of centres is parallel to SK, and of which Π$_1$, O$_1$ (here regarded as variable) are respectively a Π-point, and the mid-point of the join of the Π-points. Let α denote the angle Π$_1$SO$_1$ and β the angle SΠ$_1$O$_1$.

Then $K_1\Pi_1 = \rho/r$,

$K_1\Pi_1^2 = K_1O_1^2 + \Pi_1O_1^2$,

and $\Pi_1O_1 = \dfrac{\sin\alpha}{\sin\beta} \cdot SO_1$

Therefore $K_1O_1^2 + \dfrac{\sin^2\alpha}{\sin^2\beta} \cdot SO_1^2 = \dfrac{\rho^2}{r^2}$.

Therefore, taking SO$_1$ and SK as the axes of x and y respectively,

we have as the equation of the locus of K_1,

$$y^2 + \frac{\sin^2\alpha}{\sin^2\beta} \cdot x^2 = \frac{\rho^2}{r^2},$$

which represents an ellipse having S as centre, and a pair of conjugate diameters coincident with SO_1 and SK. By making ρ vary, we obtain a series of concentric, similar and similarly placed ellipses. Thus statement (iv) is proved.

It may be noted that in the case when $S\Pi = S\Pi'$, the lines of centres (SK, O_1K_1, O_2K_2, etc.) of the generating systems coincide, so that the web obtained is such that any point on SO_1 is the centre of a concentric system.

A coaxaloid web of the second kind may be generated in a great many different ways, of which the following one may be mentioned. In connection with Fig. 6 it was shown that if ABC remain fixed, while PQR varies so as to remain homothetic with itself with respect to S as homothetic centre, we obtain a coaxaloid system L*l*M*m*N*n* connected with ABC. If now ABC also be made to vary so as to remain homothetic with itself with respect to S as homothetic centre, we obtain a series of coaxaloid systems having the same S-point, and forming a web of the second kind; for as may easily be proved by similar triangles, their ratios are the same, their Π-points lie on two straight lines intersecting in S, and their line of centres are either parallel or, as is the case when the Π-points are the Brocard points of ABC, coincident.

Note referring to Art. 17.

Let K be the centre (Fig. 12) and TT′ the diameter through Π and Π′ of the minimum circle of a coaxaloid system whose Π-points (real) are Π and Π′. On the line of centres of the system, on opposite sides of K, take two points P and P′ such that $KP = KP' = K\Pi$. Suppose that a system of coaxal circles is constructed with TT′ as radical axis, and P, P′ as limiting points; and suppose that the radii of these are increased or diminished, as the case may be, in the ratio KT/KΠ, denoted by r, so as to give rise to a coaxaloid system with imaginary Π-points. Let K′ be the centre of any circle of this system, and K′W a radius of it parallel to any fixed direction KV. Draw K′R a radius of the circle of the

original coaxal system, whose centre is K', parallel to TT'; and draw RM perpendicular to TT', and join MP, MK'. Then if x and y be the coordinates of W, with respect to KV, KP as axes, we have

$$x^2 = K'W^2 = r^2K'R^2 = r^2(K'M^2 - RM^2).$$

But RM = PM, both being tangents from a point on the radical axis to circles of the coaxal system. Therefore

$$x^2 = r^2(K'M^2 - PM^2) = r^2(K'K^2 - PK^2) = (y^2 - K\Pi^2)KT^2/K\Pi^2.$$

Hence the equation of the locus of W is

$$y^2/K\Pi^2 - x^2/KT^2 = 1.$$

It will be noticed that the hyperbolas in this case are the conjugates of those associated with the coaxaloid system whose Π-points (real) are Π and Π', and whose ratio is also KT/KΠ.

It may be remarked that the foregoing proof and that of Art. 17 implicitly contain the following theorem which I have not seen stated elsewhere: If K'W (Fig. 5) be a variable line of fixed direction, such that its extremity K' moves on a fixed line (KK'') and such that K'W/K'Π, where Π is a fixed point, is constant, the locus of W is a hyperbola.

Systems of Conics connected with the Triangle.

By J. A. Third, M.A.

[The references in square brackets are to the articles of my paper on Systems of Circles analogous to Tucker Circles.]

I.

Systems of Six-point Conics.

1. It has been shown [4] that if two triangles ABC and PQR (Fig. 6) be in perspective with respect to any point S as centre of perspective, the sides of PQR cut the non-corresponding sides of ABC in three pairs of points which lie on a circle, provided that the angles made by the sides of PQR with those of ABC possess certain

values dependent on the position of S. In the general case, when these angles do not conform to the condition referred to, it follows from Pascal's theorem that the six points determined by the sides of PQR on the sides of ABC, lie on a conic.

Hence it follows that if any straight line be taken meeting QR, RP, PQ in D, E, F respectively (Fig. 16), and a triangle P'Q'R' be drawn in perspective with PQR with respect to S as centre and DEF as axis of perspective, the sides of P'Q'R' meet the non-corresponding sides of ABC in three pairs of points which also lie on a conic; for P'Q'R' is in perspective with ABC with respect to S as centre of perspective. By causing P'Q'R' to vary so as to remain in perspective with PQR with respect to S as centre of perspective and DEF as axis of perspective, we obtain a system of six-point conics connected with ABC.

If we project this system so that one of the conics becomes a circle and the line DEF becomes the line at infinity, then the sides of the variable triangle P'Q'R' become parallel to the corresponding sides of PQR, and consequently [3] the other conics of the system also become circles. That is, the system of conics may be projected into a coaxaloid system of circles.

We thus obtain the following propositions.

(i) The conics of the system intersect on DEF in two (real or imaginary) points, corresponding to the circular points at infinity of the system of circles.

(ii) The envelope of the system is a conic touching the sides of ABC at the points where these are met by AS, BS, CS [11, 12].

(iii) The poles of DEF with respect to the envelope and the conics of the system are collinear. This corresponds to the property of the collinearity of the centres of the coaxaloid system of circles and their envelope [8, 11]. Generally the poles of any line, with respect to the conics, and consequently their centres lie on a conic [26].

(iv) If from any point on DEF a line be drawn to the pole of DEF with respect to a variable conic of the system, the locus of the points where this line meets the conic is another conic; for parallel diameters of the corresponding coaxaloid system of circles have their extremities lying on a conic [16]. By varying the point on DEF, we obtain a system of conics associated with the original system,

and having the same envelope [17]. The poles of DEF with respect to the associated conics coincide with the pole of DEF with respect to the envelope; for in the corresponding coaxaloid system of circles, the associated hyperbolas are concentric with the envelope [16].

It may readily be proved that the associated conics are, in general, hyperbolas. In the general case DEF does not touch the conics of the original system, so that the line drawn from any point (G) on DEF through the pole (H) of DEF with respect to a variable conic of the system, meets this conic in two points (W, W') which never coincide except in the cases noted below. Hence the conic on which W, W' move is a hyperbola, W and W' lying on opposite branches.*

Degenerate forms of the hyperbolas are obtained (1) when DEF touches the conics, for then H and consequently W and W', for all positions of G and of the variable conic, coincide in the common point of contact, (2) when the conics touch each other at the points (J, J') where they meet DEF (*see next article*), for then H is the same for all the conics, and consequently for any given position of G, and for all positions of the variable conic, W and W' are collinear, and (3) when G coincides with either J or J', in which case W and W' are, for all positions of the variable conic, coincident at either J or J'.

(v) The six points in which a variable conic of the system meets the sides of ABC determine two triangles inscribed in ABC (corresponding to the triangles of constant species in the case of the circles [7]), which are such that the sides about any angle of either together with the lines joining the vertex of the angle to the two points where the conic meets DEF, form a pencil whose cross-ratio is a constant.

(vi) The conics of the system are related to any other triangle circumscribed to their envelope, in the same way that they are related to ABC [13].

2. When the original conic of a system such as has been described in Art. 1 is inscribed in ABC, the system may obviously

* If the system be regarded as the projection of a coaxaloid system of circles having imaginary II-points, in which case it is not connected with any real triangle as a system of six-point conics, W and W' lie on the same branch.

be projected into a system of coaxaloid circles one of which is inscribed in a triangle, *i.e.*, into one of the four concentric systems [22]. Hence, in this case, the conics have double contact with each other (real or imaginary) on DEF, and the pole of DEF with respect to every conic of the system is the same.

3. When DEF is made to move off to infinity while ABC and PQR, and consequently the original conic of the system, remain unaltered, *i.e.*, when the sides of the variable triangle P'Q'R' are drawn parallel to the sides of PQR, the conics of the system, having two common points at infinity (Art. 1, i), are similar and similarly placed. In this case the poles of DEF, with respect to the envelope and the conics of the system, become the centres of the envelope and the conics. Hence the centres of the conics and their envelope are collinear; and systems of parallel diameters of the conics have their extremities lying on a system of hyperbolas which have the same envelope as the original conics and are concentric with this envelope. This case is, therefore, very closely analogous to that of coaxaloid circles.

As an illustration of the theorem of the present article, the following may be given: If two transversals which intersect in K, meet the sides BC, CA, AB of a triangle in L, M, N and l, m, n respectively, and if S be the centre of perspective of ABC and the triangle determined by Mn, Nl, Lm, and if variable parallels M'n', N'l', L'm' be drawn to Mn, Nl, Lm intersecting correspondingly on CS, AS, BS, and meeting BC, CA, AB in the pairs of points L' and l', M' and m', N' and n' respectively, then the six variable points L', l', M', m', N', n' determine a system of hyperbolas whose asymptotes are parallel to the two transversals, and whose centres lie on a line through K. In this case the line-pair formed by the transversals constitutes a hyperbola similar and similarly placed to the others of the system.

When the original conic of a system is inscribed in the triangle, and DEF is at infinity, the conics, besides being similar and similarly placed, are concentric (Art. 2). Thus, if the original conic is a parabola touching the sides of the triangle, and DEF be at infinity, the other conics are coaxial parabolas.

4. Since DEF can occupy any position in the plane, every conic

which meets the sides of a triangle in six points, belongs to an infinite number of systems such as have been described in Art. 1. Every conic which meets the sides of a triangle in six points gives rise, just as in the case of the circle [2], to four S-points. Consequently, each of the above-mentioned systems has as S-point one or other of four points, and as envelope one or other of four conics touching the sides of the triangle at the points where these are met by the connectors of the opposite vertices with S. Also there are four systems for every position of DEF.

Hence we obtain the following theorems.

(i) A conic which meets the sides of a triangle in six points has, in general, double contact (real or imaginary) with four conics inscribed in the triangle.

(ii) If a conic touches the sides BC, CA, AB of a triangle at X, Y, Z respectively, and if S be the point of concurrence of AX, BY, CZ, the sides of every triangle in perspective with ABC with respect to S as centre of perspective, meet the non-corresponding sides of ABC in pairs of points which lie on a conic having double contact (real or imaginary) with the original conic. This is evident from the fact that each of the secondary conics belongs to a system having double contact with the original conic.

The converse of the immediately preceding theorem may be thus stated: If a conic κ touch the sides BC, CA, AB of a triangle at X, Y, Z, and S be the point of concurrence of AX, BY, CZ, and if a variable six-point conic κ' have double contact with κ, then two of the eight variable triangles formed by joining the six points where κ' meets the sides of ABC from side to side in pairs, is in perspective with ABC with respect to S as centre of perspective.

This leads to the following two interesting particular cases.

(a) If ABC be a triangle circumscribed to a conic, and two variable tangents be drawn to the conic meeting BC, CA, AB, one in L, M, N and the other in l, m, n respectively, then the variable triangle determined by the connectors Mn, Nl, Lm (or mN, nL, lM) is in perspective with ABC with respect to a fixed point S as centre of perspective, S being the point of concurrence of the lines joining A, B, C to the points of contact of the conic with the opposite sides. In this case a variable line-pair is taken instead of the variable conic of the general case.

(*b*) If ABC be a triangle circumscribed to a parabola, and a variable tangent be drawn to the parabola meeting BC, CA, AB in L, M, N respectively, and if through L, M, N parallels be drawn to CA, AB, BC (or AB, BC, CA) respectively, the variable triangle so determined is in perspective with ABC with respect to a fixed point S as centre of perspective, S being the point of concurrence of the lines joining A, B, C to the points of contact of the parabola with opposite sides. In this case, one of the variable tangents of (*a*) is the tangent at infinity of the parabola. The present case could be established independently of (*a*) by recollecting that the Simson lines of a point, with the line at infinity, form a coaxaloid system of circles [7], and have, therefore, a common S-point. (*a*) could then be deduced from (*b*) by projection.

II.

Systems of Six-tangent Conics.

5. By Brianchon's theorem if two triangles ABC and PQR be in perspective with respect to any line *s* as axis of perspective, the vertices of PQR connect with the non-corresponding vertices of ABC by means of six lines which touch a conic. Hence if any point U be taken in the plane, and a triangle P'Q'R' be drawn in perspective with PQR with respect to *s* as axis and U as centre of perspective, the vertices of P'Q'R' connect with the non-corresponding vertices of ABC by means of six lines which also touch a conic. By causing P'Q'R' to vary so as to remain in perspective with PQR with respect to *s* as axis and U as centre of perspective, we obtain a system of six-tangent conics connected with ABC.

The general properties of such a system are most readily obtained by observing that it is the polar reciprocal of a six-point system such as is discussed in Art. 1, *s* and U of the former corresponding to S and DEF of the latter. We thus obtain the following propositions.

(i) The conics of the system have a pair of common tangents (real or imaginary) which meet in U.

(ii) The envelope of the system is a conic circumscribed to ABC,

and touching at its vertices the lines connecting these with the points where the opposite sides meet s.

(iii) The polars of U with respect to the envelope and the conics of the system are concurrent; generally the polars of any point with respect to the conics envelop a conic.

(iv) The envelope of the tangents to a variable conic of the system from the point where any line through U meets the polar of U with respect to the conic, is a conic. By varying the line through U, we obtain a system of conics associated with the original system and having the same envelope.

(v) The six tangents from the vertices of ABC to a variable conic of the system determine two triangles circumscribed to ABC which are such that the extremities of any side of either together with the points of intersection of that side and the tangents to the conic from U, form a range whose cross-ratio is a constant.

(vi) The conics of the system are related to any other triangle inscribed in their envelope in the same way that they are related to ABC.

To these may be added:—

(vii) The poles of any line with respect to the conics, and consequently their centres, lie on a conic [25].

6. When one of the conics, say the original conic, of a system such as has been described in the preceding article, is circumscribed to ABC, the system is obviously the polar reciprocal of a system of the kind referred to in Art. 2. Hence, in this case, the conics touch their two common tangents at the same pair of points, and the polar of U with respect to every conic of the system is the same.

7. If U be a focus of one conic of a system, say the original conic, it is a focus of all the conics of the system. This is evident if we reciprocate a coaxaloid system of circles with respect to U as centre of reciprocation. In this case, since the centres of the circles reciprocate into the directrices corresponding to U of the conics, or by Art. 1, (iii), the directrices corresponding to the common focus are concurrent.

When U is taken so as to coincide with one of the II-points,

say Π, of the reciprocal system of coaxaloid circles (Fig. 9), the eccentricity of the conic into which the circle K′ reciprocates is

$$\text{K′Π/ radius of circle K′}$$

$$= \text{K′Π}/(\text{K′Π} \cdot \text{KT}/\text{KΠ}) = \text{KΠ}/\text{KT} = \text{a constant.}$$

Therefore, in this case, the conics into which the circles reciprocate, have the same eccentricity, *i.e.*, they are similar. Also, since the centre of reciprocation U is a focus of the envelope of the circles, the envelope of the conics is, in general, a circle.

Hence we have the following theorem: If a system of similar conics have a common focus U and concurrent directrices, their envelope is, in general, a circle; and if a triangle be inscribed in this circle, and s be the line on which the tangents to the circle at the vertices meet the opposite sides, the tangents from the vertices to a variable conic of the system determine by their intersection two variable triangles which are always in perspective with themselves with respect to s and U as axis and centre of perspective.

In order to determine the centre and radius of the circle enveloping a system of similar conics having a common focus and concurrent directrices we may proceed as follows. Let U be the common focus, O the point of concurrence of the directrices, d the perpendicular through O to UO, and A, A′ the vertices, lying on UO, of the conic of the system, denoted by κ, whose directrix is d. Then since d is the reciprocal, with respect to U, of the centre of the auxiliary circle of the envelope of the reciprocal system of coaxaloid circles, κ is the reciprocal of this circle. Again, since O is the reciprocal of the line of centres of the reciprocal system of coaxaloid circles, and U (being the centre of reciprocation) the reciprocal of the line at infinity, UO is the reciprocal of the point at infinity on the line of centres of the coaxaloid circles, *i.e.*, of the point of concurrence of the line of centres and the two tangents common to the envelope of the coaxaloid circles and its auxiliary circle (the tangents at T, T′ in Fig. 9). Therefore A and A′, where κ meets UO, are the reciprocals of these tangents, and are, consequently, the common points of κ and the circle enveloping the conics. Therefore AA′ is a diameter of this circle, whose centre and radius, are, consequently, determined.

In the particular case when the conics are parabolas, one of the vertices A, A′, say A′, is at infinity; the enveloping circle, con-

sequently, degenerates into the line at infinity and the tangent at A to the parabola whose directrix is d. Otherwise: If a system of parabolas have a common focus and concurrent directrices, they have a common tangent (real) in addition to the line at infinity.

If one conic of a system, say the original conic, be circumscribed to the triangle ABC, and U be its focus, then the conics, besides having a common focus, have a common directrix. This result is obtained by reciprocating a concentric system of coaxaloid circles connected with a triangle, from any centre (U). Hence we have the theorem: If a conic be circumscribed to a triangle, U a focus, d the corresponding directrix, and s the line on which the tangents at the vertices meet the opposite sides, and if a system of conics be constructed having the same focus U and directrix d, the tangents from the vertices of the triangle to a variable conic of this system determine two variable triangles which remain in perspective with themselves with respect to s and U as axis and centre of perspective respectively.

In the particular case when the conic circumscribed to ABC is the circumcircle and U the circumcentre, then the common directrix d is the line at infinity, and hence the other conics are circles concentric with the circumcircle. The theorem involved in this case could also be obtained by elementary geometry.

8. By reciprocating the theorems of Art. 4 we could obtain an analogous set of theorems applying to six-tangent conics connected with the triangle. I shall limit myself to enunciating two of these.

(i) If a conic be circumscribed to a triangle ABC, and s be the line on which the tangents at the vertices meet the opposite sides, then the vertices of every triangle in perspective with ABC with respect to s as axis of perspective, connect with the non-corresponding vertices of ABC by pairs of lines which touch a conic having double contact (real or imaginary) with the original conic (Art. 4, ii).

(ii) If ABC be a triangle inscribed in a conic, and if two variable points be taken on the conic, connecting with A, B, C, the one by the lines L, M, N and the other by the lines l, m, n respectively, then the variable triangle determined by the points of intersection of the pairs M and n, N and l, L and m (or m and N, n and L, l and M) is in perspective with ABC with respect to a fixed line s as axis of perspective, s being the line on which the tangents to the conic at A, B, C meet the opposite sides.

Systems of Spheres connected with the Tetrahedron.

By J. A THIRD, M.A.

[References in square brackets are to my paper on Systems of Circles analogous to Tucker Circles.]

1. If A_1 and B_3 (Fig. 17), B_1 and C_3, C_1 and D_3, D_1 and A_3, A_2 and C_2, B_2 and D_2 be pairs of points on the edges

AB, BC, CD, DA, AC and BD

respectively of a tetrahedron ABCD, such that the two points on any edge are concyclic with the two points on any other edge—a manifestly possible condition of things—then the twelve points lie on a sphere.

Demonstration. Since $A_1A_2B_3C_2$, $A_2A_3C_2D_1$, and $A_3A_1B_3D_1$ are concyclic quartets, it is obvious that the six points $A_1A_2A_3B_3C_2D_1$ lie on a sphere cutting the edges of the trihedral angle A in three pairs of points. Again since $A_1A_2B_3C_2$, $B_1B_3C_3A_1$, and $C_2C_3A_2B_1$ are quartets of concyclic points, the six points $A_1A_2B_1B_3C_2C_3$ are concyclic [1]. Therefore B_1 and C_3 lie on the same section of the sphere $A_1A_2A_3B_3C_2D_1$ as the quartet $A_1A_2B_3C_2$. Similarly it might be shown that C_1 and D_3 lie on the same section of this sphere as the quartet $A_2A_3C_2D_1$ and that B_2 and D_2 lie on the same section as the quartet $A_3A_1B_3D_1$. Therefore all the twelve points lie on the same sphere.

The foregoing theorem is an easy extension of that given in Art. 1 of my paper on *Systems of Circles analogous to Tucker Circles.* It is obvious from the demonstration given that it might be enunciated also in the following way: If two points be taken on each edge of a tetrahedron, and if (i) the six points lying on the three edges of each solid angle lie on a sphere, or (ii) the six points lying on each face be concyclic, then all the twelve points lie on a sphere.

2. In the figure of the preceding article (Fig. 17) let S_1 be the centre of perspective of the triangle ABC and the triangle determined by the connectors A_1A_2, B_1B_2, C_1C_2; S_2 the centre of

perspective of the triangle BCD and the triangle determined by the connectors B_1B_2, C_1C_3, D_2D_3; and S_3, S_4 the corresponding points on the faces CDA and DAB respectively. Also let P be the point of intersection of B_1B_3 and C_2C_3.

Since $B_1C_2B_3C_2$, $B_3C_2B_2C_1$, and $B_2C_1B_1C_3$ are quartets of concyclic points, it follows, by an obvious extension of the principle of the concurrency of the three radical axes of three circles taken in pairs, that the connectors B_3C_2 and B_2C_1 meet in a point on BC, say Y. Again if AS_1 meets BC in X_1, since the opposite sides of the complete quadrangle AB_3PC_2 meet the transversal BC in three pairs of points in involution, $X_1YBC_3B_1C$ form an involution. Similarly if DS_2 meets BC in X'_1 it is evident that $X'_1YBC_3B_1C$ form an involution. Therefore X'_1 is the same point as X_1.

Similarly it might be shown that AS_3, BS_2 meet on CD, say in X_2; that BS_4, CS_3 meet on DA, say in X_3; that CS_1, DS_4 meet on AB, say in X_4; that BS_1, DS_3 meet on AC, say in X_5; and that AS_4, CS_2 meet on BD, say in X_6.

The planes ADX_1, BCX_3 obviously intersect on the line X_1X_3; the planes ABX_2, CDX_4 on the line X_2X_4; and the planes BDX_5, ACX_6 on the line X_5X_6.

The lines X_1X_3, X_2X_4, X_5X_6 are concurrent. This may be proved as follows: From the harmonic properties of quadrilaterals (the quadrilateral $BX_1S_1X_4$ being considered) it is evident that the connector X_1X_4 meets AC in a point which is the harmonic conjugate of X_5 with respect to A and C. Similarly it is evident that the connector X_2X_3 meets AC in a point which is the harmonic conjugate of X_5 with respect to A and C. Thus X_1X_4 and X_2X_3 meet in a point. Therefore X_1, X_2, X_3, X_4 lie in a plane. Therefore X_1X_3 and X_2X_4 meet in a point. Similarly it may be shown that X_2X_4 and X_5X_6 meet in a point, and also X_5X_6 and X_1X_3. Therefore, unless the six points X_1, X_2, X_3, X_4, X_5, X_6 lie in one plane, which is impossible, X_1X_3, X_2X_4, and X_5X_6 meet in the same point.

Hence the six planes (ADX_1, etc.) determined by the six points X_1, X_2, X_3, X_4, X_5, X_6 and the opposite edges of the tetrahedron have a common point, say Σ. These planes may be called for convenience the Σ-planes. They are, of course, determinate when the point Σ is given.

The twelve points A_1, B_1, C_1, etc., in which the sphere meets

the edges of the tetrahedron, may be regarded as determined by four planes $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$ and $D_1D_2D_3$ which cut off the solid angles A, B, C, D respectively (thus determining an octahedron inscribed in the sphere). These four planes, which are analogous to the directive chords of a coaxaloid system of circles [3] intersect two and two on the six Σ-planes. This may be demonstrated as follows: B_1B_3 and C_2C_3 intersect on AX_1; and B_1B_2 and C_1C_3 intersect on DX_1. Therefore the planes $B_1B_2B_3$ and $C_1C_2C_3$ intersect on the plane ADX_1, *i.e.*, $AD\Sigma$. Similarly it might be shown that each of the other five pairs in which the four planes $A_1A_2A_3$, etc., may be taken intersect on one of the five remaining Σ-planes (a different plane for each pair). Thus the four planes $A_1A_2A_3$, etc., determine a tetrahedron, such that its edges lie with the edges of the original tetrahedron ABCD in six planes which meet in the point Σ.

3. From the results established in the preceding article it is readily deducible that the following construction is possible, namely, to draw other four planes parallel and corresponding as regards the intersection of the edges, to $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$ and $D_1D_2D_3$, so as to intersect two and two on the same Σ-planes as their correspondents; for if the vertices of the tetrahedron formed by the four planes $A_1A_2A_3$, etc., be a, b, c, d lying on the lines ΣA, ΣB, ΣC, ΣD respectively, it is clearly possible to find any number of quartets of corresponding points a', b', c', d' lying on the same lines such that the ratios $\Sigma a/\Sigma a'$, $\Sigma b/\Sigma b'$, $\Sigma c/\Sigma c'$ and $\Sigma d/\Sigma d'$ shall be equal.

If now the construction indicated be made, the twelve points determined by the four new planes on the edges of the tetrahedron also lie on a sphere.

Demonstration. Three of the new planes determine on the face ABC three lines. These lines are parallel to A_1A_2, B_1B_3, C_2C_3 and intersect two and two in the same manner on CS_1, AS_1, BS_1. Therefore [3] the six points which they determine on the edges of the face ABC, are concyclic. Similar statements apply to the other faces. Therefore by Art. 1, the twelve points which the four new planes determine on the edges of the tetrahedron lie on a sphere.

By drawing a system of parallel planes in the manner indicated, we obtain a system of twelve-point spheres connected with the tetrahedron.

4. The centres of the spheres of such a system are collinear.

Demonstration. Consider three spheres of the system. Let M_1, M_2, M_3 (Fig. 18) be the centres, known to be collinear [8], of the three circular sections of these made by one face of the tetrahedron, and K_1, K_2, K_3 the centres, also collinear, of the corresponding sections made by another face. Perpendiculars drawn from M_1, M_2, M_3 to the face in which they lie are coplanar, and meet the corresponding perpendiculars from K_1, K_2, K_3, also coplanar, in three points, Γ_1, Γ_2, Γ_3 which are the centres of the spheres. Γ_1, Γ_2, Γ_3 lie on the common section of the planes determined by the two triads of perpendiculars, and are therefore collinear.

5. The following properties are easy inferences from the corresponding properties of coaxaloid circles connected with the triangle.

(i) If a system of parallel planes be drawn through the centres of the spheres of a system such as has been described, the circumferences of the great circles thus determined lie on a hyperboloid of one sheet.

This is evident from the following considerations. When the diametral planes are perpendicular to a face of the tetrahedron, the great circles pass through the extremities, known to lie on different branches of a hyperbola [16], of a system of parallel diameters of the coaxaloid system of circles on that face. Hence, since their centres are collinear, the great circles, in this case, lie on a hyperboloid of one sheet. Hence the extremities of any system of parallel diameters of these great circles lie on opposite branches of a hyperbola. Therefore each of these diameters is to the distance of its mid-point from a certain fixed point in a constant ratio. If now these parallel diameters be turned round their mid-points through any angle in the plane in which they lie, the ratio referred to is unaltered and therefore [*note*] their extremities lie on opposite branches of a hyperbola. This proves the proposition.

By varying the direction of the diametral planes a system of hyperboloids associated with the spheres is obtained.

(ii) Since the circumferences of a system of parallel great circles of the spheres lie on a hyperboloid of one sheet, these circles would, if turned round their centres into one plane, form a coaxaloid

system. Hence the spheres also form a coaxaloid system, *i.e.*, they can be obtained from a coaxal system of the intersecting species by increase or diminution, in a constant ratio, of the radii of the latter.

(iii) If the spheres be cut by any plane through their line of centres, the great circles so obtained are, as we have seen, coaxaloid; hence their envelope and that of their associated hyperbolas is a conic. By rotating this plane round the line of centres as axis we see that the envelope of the spheres and their associated hyperboloids is a quadric of revolution.

This quadric touches the edges of the tetrahedron at

$$X_1,\ X_2,\ X_3,\ X_4,\ X_5,\ X_6;$$

for the envelopes of the coaxaloid systems of circles on the faces of the tetrahedron do so, and these envelopes are obviously sections of the quadric.

(iv) The spheres of the system are related to any other tetrahedron whose edges touch their envelope, in the same way that they are related to ABCD.

"La Perspective d'une Conique est une Conique." Démonstration Élémentaire.

By M. L. Leau.

Edinburgh Mathematical Society.

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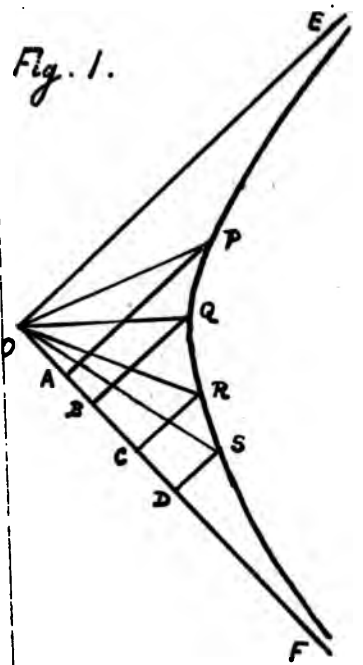

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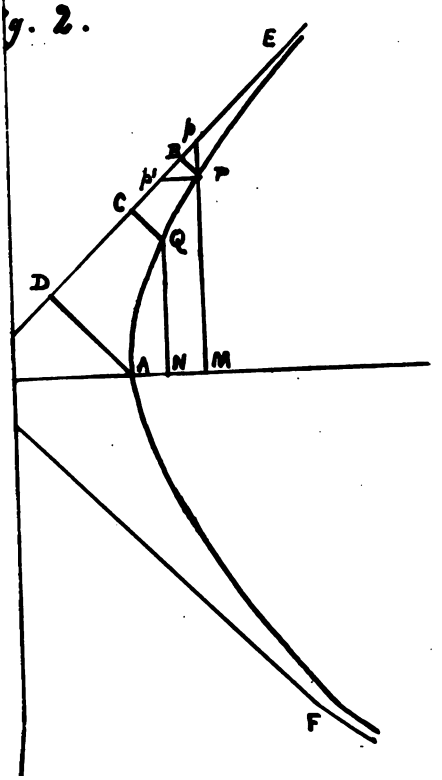

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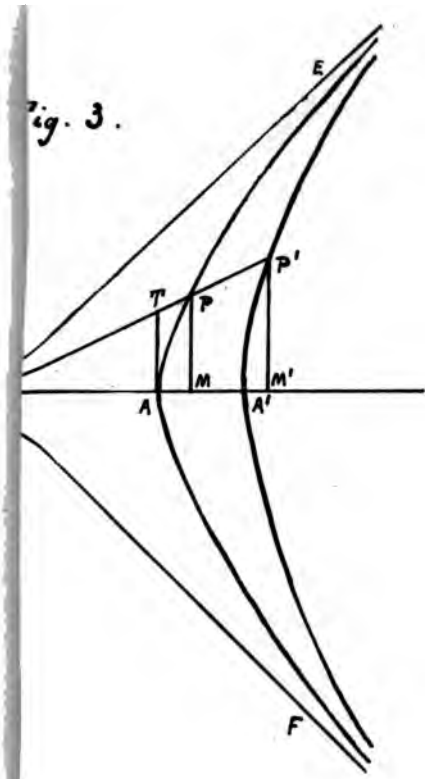

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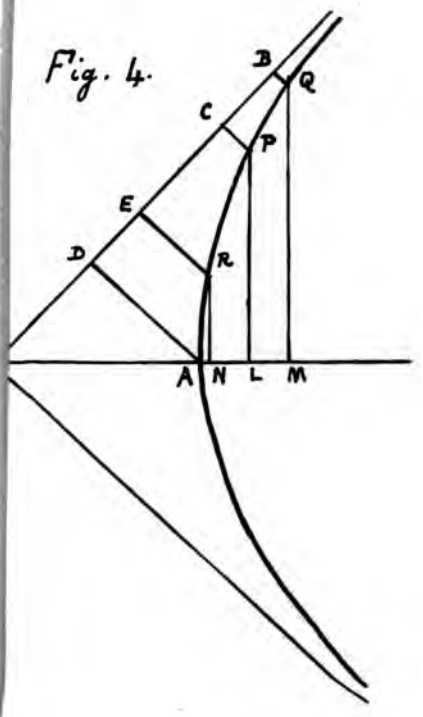

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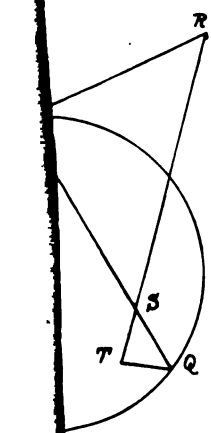

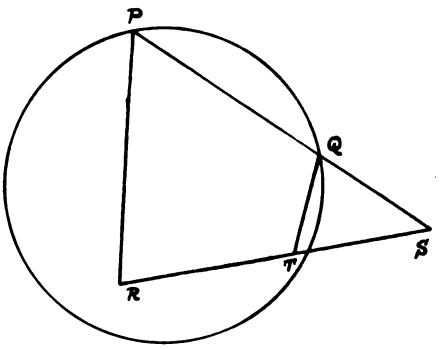

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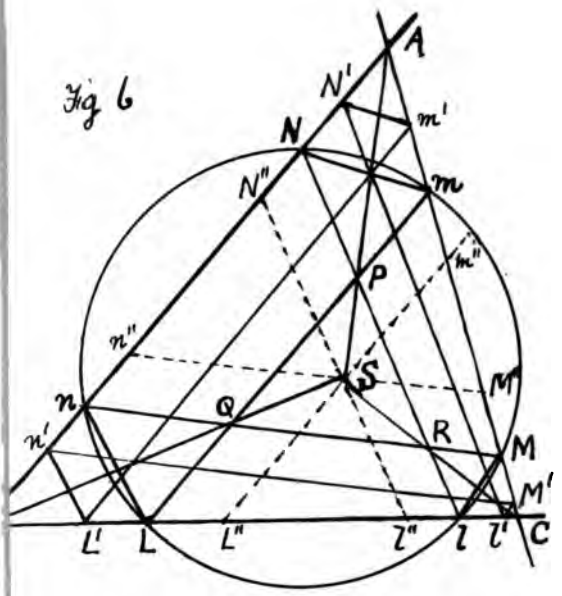

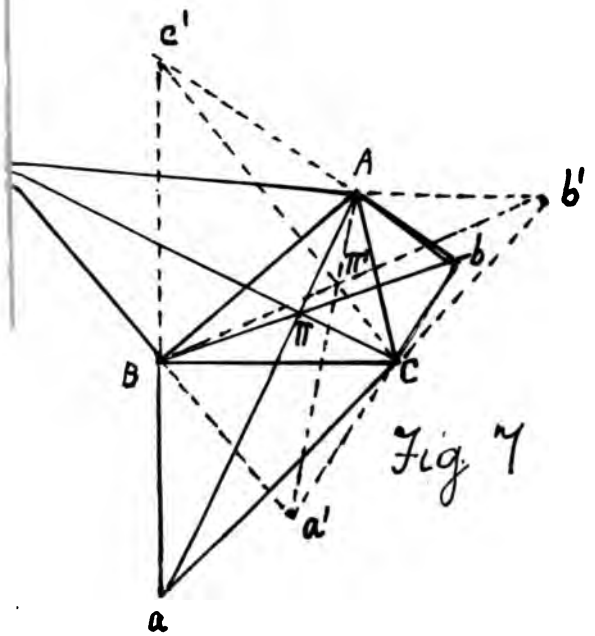

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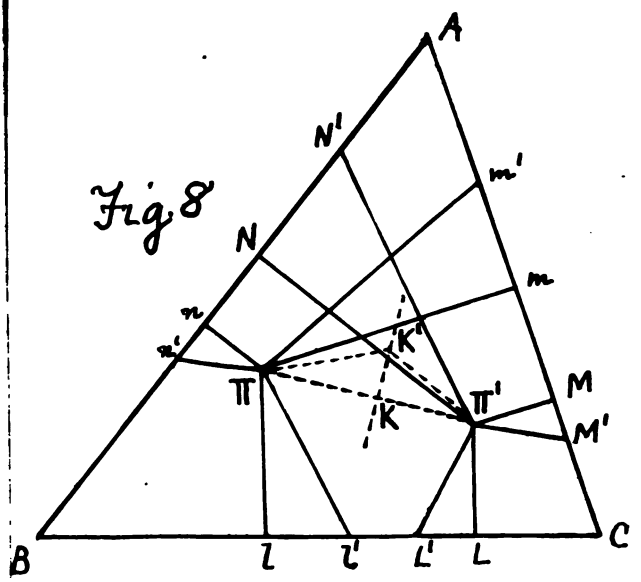

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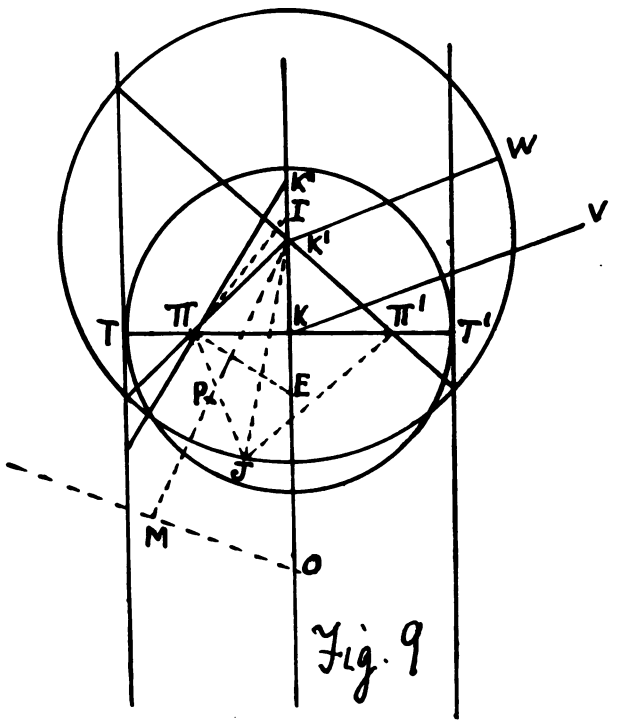

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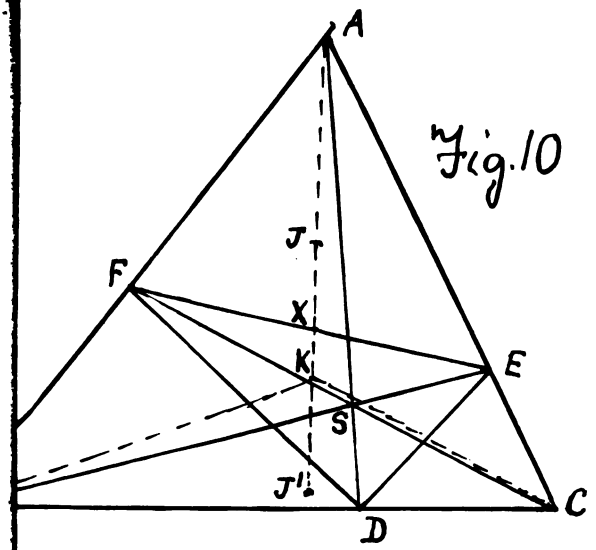

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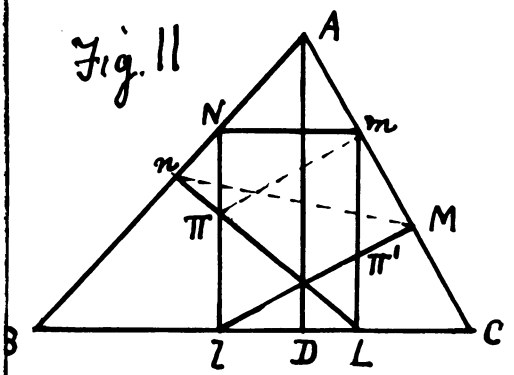

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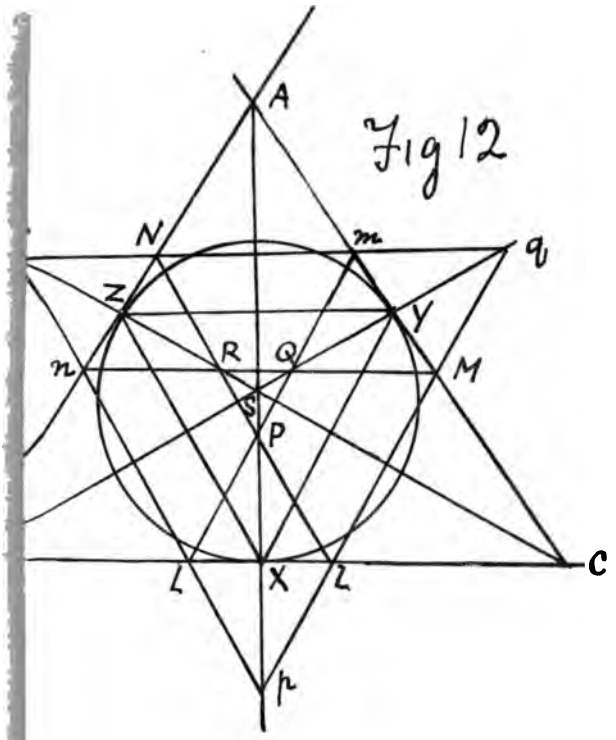

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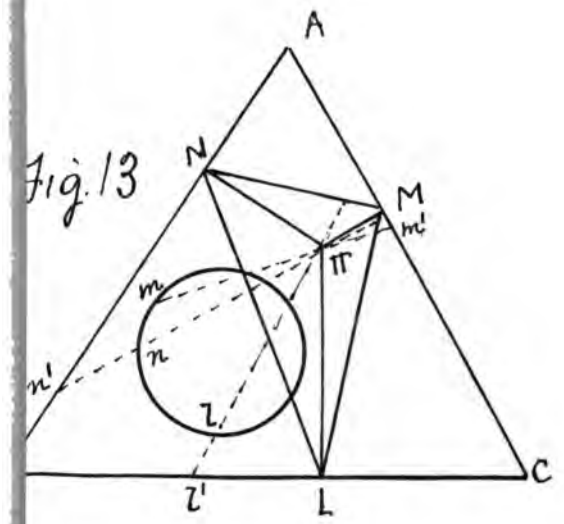

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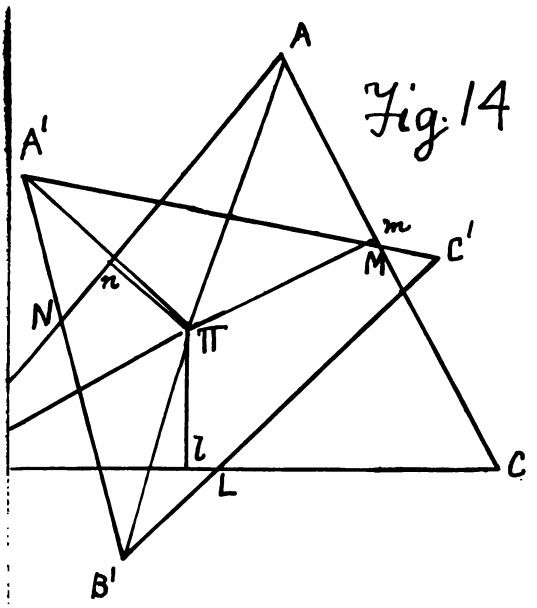

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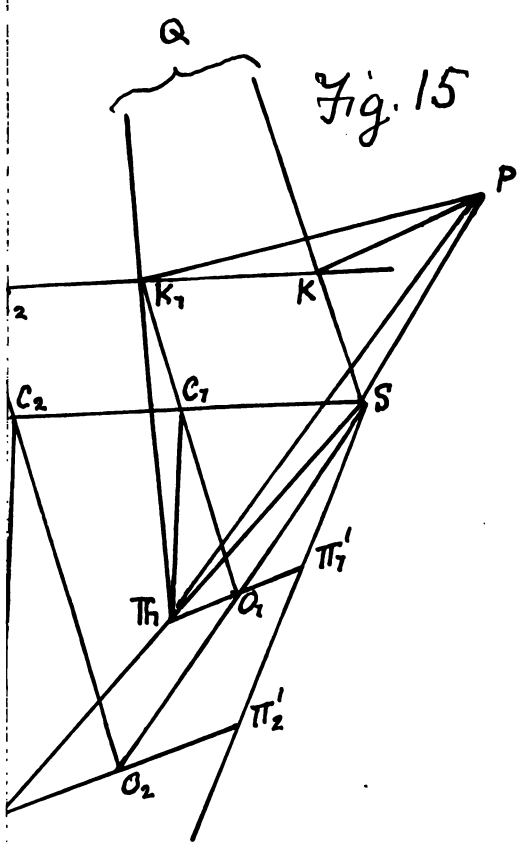

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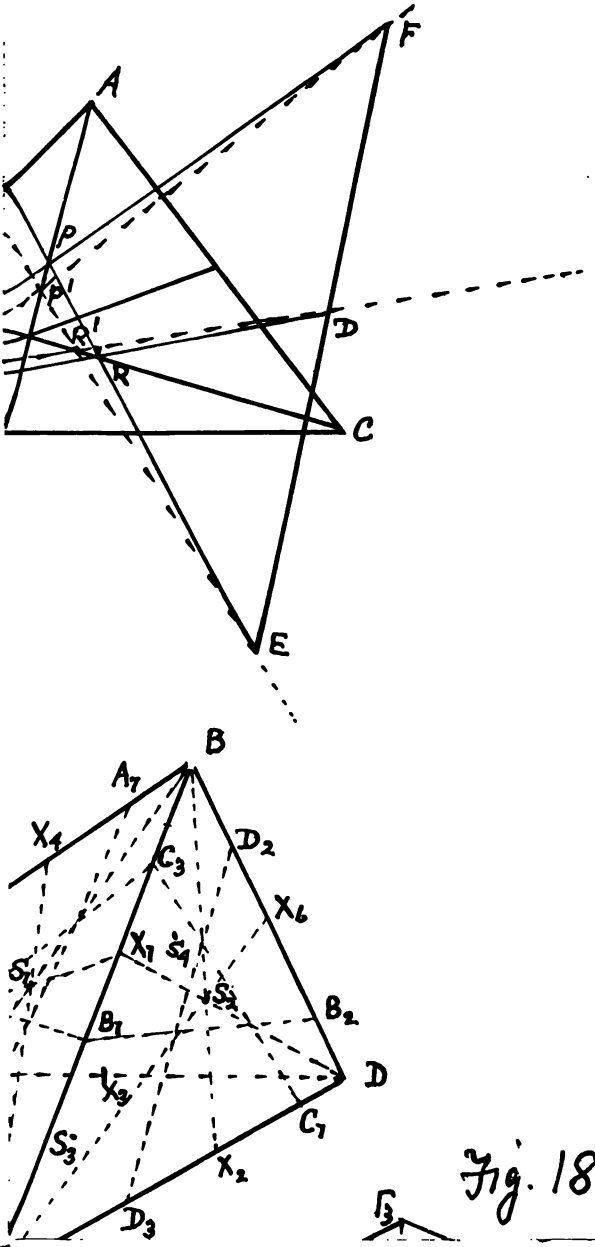

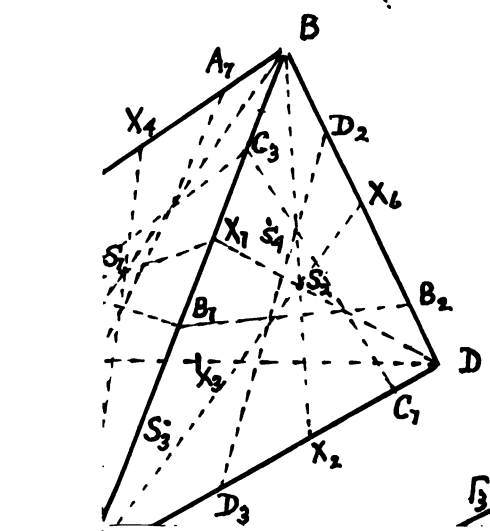

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PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

EIGHTEENTH SESSION, 1899–1900.

First Meeting, 10th November, 1899.

Dr MACKAY in the Chair.

For this Session the following Office-bearers were elected :—

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Mr JAMES MITCHELL, M.A., B.Sc.;
Mr JOHN A. THIRD, M.A., D.Sc.

Elementary Proof that the Arithmetic Mean of any number of Positive Quantities is greater than the Geometric Mean.*

By G. E. Crawford, M.A.

Def. The arithmetic mean of n quantities $a, b, c, d \ldots$ is $(a+b+c+d \ldots)/n$.

Def. The geometric mean of n quantities $a, b, c, d \ldots$ is $(abcd \ldots)^{1/n}$.

Lemma 1. Both the arithmetic mean and the geometric mean are intermediate in value between the greatest and the least of the n quantities, if they be all positive.

For, if a be the greatest, and b the least,

$$\begin{aligned}
\text{the arithmetic mean} &= (a+b+c+d \ldots)/n \\
&< (a+a+ \ldots \text{ to } n \text{ terms})/n \\
&< na/n \\
\textit{i.e.,} \quad &< a .
\end{aligned}$$

$$\begin{aligned}
\text{and the arithmetic mean} &> (b+b \ldots \text{ to } n \text{ terms})/n \\
&> nb/n \\
\textit{i.e.,} \quad &> b .
\end{aligned}$$

Similarly,

$$\begin{aligned}
\text{the geometric mean} &= (abc \ldots)^{1/n} \\
&< (aa \ldots \text{ to } n \text{ factors})^{1/n} \\
&< (a^n)^{1/n} \\
\textit{i.e.,} \quad &< a .
\end{aligned}$$

$$\begin{aligned}
\text{and the geometric mean} &> (bbb \ldots \text{ to } n \text{ factors})^{1/n} \\
&> (b^n)^{1/n} \\
\textit{i.e.,} \quad &> b .
\end{aligned}$$

Q.E.D.

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* The above proof is a modification of the elegant proof given by Dr G. H. Bryan in his *Middle Algebra*, and was obviously suggested by it. No third proof on the same lines can be given. Both have this logical advantage referred to by Dr Bryan, that the number of mental steps in the process is finite G. E. C.

Lemma 2. If two positive quantities a and b have the same sum as two other positive quantities x and y, then the greatest and least of the above four quantities occur together in the same pair, and their product is less than that of the other two quantities, which are intermediate to them.

For, say a is the greatest of the four, then since $a+b=x+y$, we have $a-x=y-b$.

But a is $>x$, $\therefore$ y is $>b$.

Again $a-y=x-b$.

But a is $>y$, $\therefore$ x is $>b$.

Hence both x and y are $>b$;

$\therefore$ b is the least of the four quantities, and it occurs in the same pair with a which is the greatest.

Also
$$\begin{aligned} 4xy &= (x+y)^2-(x-y)^2 \\ &= (a+b)^2-(x-y)^2 \\ &> (a+b)^2-(a-b)^2, \text{ since } a-b \text{ is } > x-y, \\ &> 4ab. \end{aligned}$$

$\therefore$ xy is $>ab$.

Lemma 3. If one of n quantities be identical with the arithmetic mean of all the n, then the remaining $n-1$ quantities have also the same arithmetic mean.

For let A, b, c, d . . be the n quantities of which the first, A, is also the arithmetic mean of all,

$$\therefore \quad A+b+c+d\ldots = nA;$$
$$\therefore \quad b+c+d\ldots = (n-1)A;$$
$$\therefore \quad (b+c+d\ldots)/(n-1) = A. \qquad \text{Q.E.D.}$$

Prop. Now let A be the arithmetic mean of n positive quantities a, b, c, $d\ldots$ of which a is the greatest and b the least.

$\therefore$ A is intermediate between a and b (Lemma 1).

Choose x so that $A+x=a+b$.

Then A is not the least of the four quantities A, x, a, b, nor is it the greatest;

$\therefore$ A and x are both intermediate to a and b, which are respectively the greatest and least of the four, (Lemma 2), and . $Ax>ab$

Also the 3 sets of quantities:

$$a,\ b,\ c,\ \mathrm{d}\ \ldots\ (n \text{ in number}),$$
$$\mathrm{A},\ x,\ c,\ d\ \ldots\ (\text{„ „ „ }),$$
$$x,\ c,\ d\ \ldots\ (n-1 \text{ in number}),$$

all agree in having the same arithmetic mean, viz., A; the first two sets because $a+b=\mathrm{A}+x$, and the last two sets by Lemma 3;

$\therefore\ abcd\ldots < \mathrm{A}xcd\ldots.$

$< \mathrm{A}.\mathrm{A}yd\ldots$

(by application of the same process, y playing the same part now among the $n-1$ quantities $xcd\ldots$ as x did before among the n quantities $abcd\ldots$).

Or, finally, $abcd\ldots < \mathrm{A}.\mathrm{A}.\mathrm{A}\ldots$ to n factors,

i.e., $< \mathrm{A}^n$;

$\therefore$ taking the nth root,

$$\mathrm{A} > (abcd\ldots)^{1/n};$$

$\therefore$ the arithmetic mean > the geometric mean.

Q.E.D.

The Dissection of any two Triangles into mutually similar pairs of Triangles.

By R. F. Muirhead, M.A., B.Sc.

1. The problem discussed is that of dissecting two given triangles into triangular parts which shall consist of mutually similar pairs of triangles, so that the first given triangle A being dissected into the triangles a_1, a_2, a_3..., and the second given triangle B being dissected into the triangles b_1, b_2, b_3..., we shall have a_1 similar to b_1, a_2 to b_2, and so on.

2. A little consideration will show that in general the problem is insoluble, if the number of parts of each given triangle is restricted to two.

We shall therefore consider the case in which each of the two given triangles is to be dissected into three triangular parts.

3. The first type of solution here given, is when an internal point of each triangle is joined to each of its vertices, so as to divide it into three triangular parts. The solution is as follows:—

Figure 1.

Let ABC be one of the given triangles.

On BC, on the side opposite from ABC, describe the triangle DBC similar to the other given triangle A'B'C', and join AD.

Make $\angle BAE = \angle CAD$

and $\angle ABE = \angle ADC$.

Join CE.

Make $\angle BDF = \angle CDA$

and $\angle DBF = \angle DAC$.

Join CF.

By construction, the triangle ABE is equiangular with the triangle ADC. $\therefore$ BA : AE :: DA : AC.

$\therefore$ BA : AD :: EA : AC.

Hence, $\angle BAD$ being $= \angle EAC$, the triangles BAD, EAC are also similar to one another.

In like manner, the triangles DFB, DFC are similar to DCA and DBA respectively.
Hence DFB is similar to BEA, and DFC to CEA.

It follows at once that $\angle BEC = \angle BFC$.

Again BE : EC is compounded of BE : EA and EA : EC, *i.e.*, of DF : FB and CF : FD, $\therefore$ it is the same as CF : FB.
Hence the triangle BEC is similar to the triangle CFB, and in fact the figure BECF is a parallelogram.
Thus AEB, BEC, CEA are similar to BFD, CFB, and DFC respectively.

4. For a given pair of triangles, the solutions of this type are 9 in number, if we distinguish between triangles which are "symmetrically" or "perversely" equal to one another; for the side of ABC on which DBC is to be placed can be chosen in 3 ways, and the side of A'B'C' which is to be homologous to BC in the triangle DBC can also be chosen in 3 ways.

5. If, however, we drop the distinction between a plane figure and its perverse, *i.e.*, if we permit the triangle A'B'C' to be *turned over* on its plane, there will be 18 solutions of the type we are considering.

6. It is to be noted, however, that some of the solutions may involve negative areas. This will happen when the sum of the angles A and D is greater than two right angles. The points E and F will then lie *without* the triangles ABC, DBC, respectively; and the problem solved by the construction will in that case be this: To find two pairs of similar triangles such that one of each pair being taken, they will build up a quadrilateral, and the remaining two will also build up a quadrilateral, these quadrilaterals being decomposable into two triangles of given shape and two others which are similar to one another.

7. Note also that if ABCD be a cyclic quadrilateral, the triangles BEC, BFC collapse, so that the given triangles in that case can be dissected into *two* pairs of mutually similar triangles.

8. The second type of solution here considered is when one triangle ABC is dissected by joining an angular point A to a

point D of the opposite side BC, then joining another angular point C to a point E in the line AD, and the second given triangle is dissected in like manner. The solution may be stated thus:—

FIGURE 2.

Let ABC and A′B′C′ be the two given triangles.
Let us make $C'A'D' = B$.

$$\therefore \quad A'D'B' = B + C' \quad \text{and} \quad B'A'D' = A' - B.$$

Make $\quad CAD = A' - B, \quad$ then $\quad DAB = A - A' + B.$

Make $\quad A'C'E' = A - A' + B, \quad \therefore \quad D'C'E' = C' - A + A' - B$
$$= C - B'.$$

Make $\quad ACE = B', \quad \therefore \quad DCE = C - B' = D'C'E'.$

And $\quad EDC = B + A - A' + B = D'E'C'$
$$= B - C + C' + B'.$$

Thus the triangles ABD, ACE, CDE are similar to the triangles C′A′E′, A′B′D′, C′E′D′, respectively; the homologous sides being given in the same order in the two sets.

We may note, however, that two of the pairs are perversely similar.

9. The question may be put: How many solutions of this type exist for the dissection of a given pair of triangles? Noting that the undivided angles (B and B′ in the above dissection) may be chosen in 9 ways from the two given triangles, and that there is then a further choice of 2×2 ways of completing the construction, we see that there would seem to be 36 solutions of this type. But of these some would involve negative areas: if, *e.g.*, the angle B were greater than the angle A′ in the solution above given, AD′ would fall outside the triangle A′B′C′, as in

FIGURE 3.

The problem solved in this figure would be: "Given two triangles, to form two triangles similar to one another, each made from one of the given triangles and one of each of two other pairs of similar triangles." Thus ACE and AB′D′ are similar to one another, also CDE and C′E′D′, and then together with the given triangles ABC, A′B′C′ make up the similar triangles ABD and C′A′E′.

Out of the 36 theoretically possible solutions of the original problem, the number in which no "negative areas" occur would depend on the shapes of the given triangles, and I do not here attempt to discuss that point any further.

10. Lastly, it may be shown that solutions of the two types that have been discussed, are particular cases of a more general dissection which dissects two triangles into two sets of *four* triangles which are similar in pairs, one angle in the figure being arbitrary.

FIGURE 4.

Take ABCD as in the first case.

Draw AE'E'' making $\angle$ BAE' arbitrary, $= \theta$ say.

Make $\angle$ DBF = $\angle$ BAE',

$\angle$ DCF' = $\angle$ CAE',

$\angle$ BCE = $\angle$ CBF,

$\angle$ CBE = $\angle$ BCF,

and $\angle$ BDF' = $\angle$ ABE;

where AE'E'', BE''E, CEE' meet one another in E, E', E'',

and DF''F', BFF'', CF'F meet one another in F, F', F''.

Then $\angle$ CDF' = $\angle$ ACE, since the former, with 5 other angles, makes up the angles of the triangle BDC, while the latter, with 5 angles equal respectively to the other 5, makes up the angles of the triangle ABC.

Hence the triangles ABE'', BCE, CAE' are directly similar respectively to the triangles BDF'', CBF, DCF'. Hence obviously the triangle EE'E'' is similar perversely to the triangle FF'F''. Thus the more general problem is solved.

11. Now if we take $\theta = 0$, the triangles ABE'' and BDF'' will collapse, giving us a solution of the second type by dissection into *three* parts. The same result may also be obtained by choosing θ so that one of the other angles shall vanish, such as CBE, or BCE'.

12. Again if we choose θ so that the triangle EE'E'' shall shrink to a point, the trinangle FF'F'' will do so also, and we shall have a solution of the first type.

13. Certain properties of the Figure 4 may be noted. Taking B′, C′, A′ to denote the angles of the triangle BCD,

$$\angle BAE' = \angle DBF = \theta;$$
$$\angle E'AC = \angle DCF' = A - \theta;$$
$$\angle ACE = \angle CDF' = C - B' + \theta;$$
$$\angle ECB = \angle CBF = B' - \theta;$$
$$\angle CBE = \angle BCF = C' - A + \theta;$$
$$\angle EBA = \angle BDF' = B + A - C' - \theta = B' + A' - C - \theta.$$

The exterior angles of the triangle EE′E″ are A + A′, B + B′, C + C′, respectively, so that the shape of the triangles EE′E″ and FF′F″ is independent of θ.

The triangles BCE, CBF are equal.

The triangles BDF″, ABE″ are in the duplicate ratio of DB : BA.

„ „ CDF′, CAE′ „ „ „ „ „ „ DC : CA.

„ „ EE′E″, FF′F″ „ „ „ „ „ „ E′E″ : F′F″.

The ratio E′E″ : F′F″ is expressible as a somewhat complicated function of θ and the angles of the two triangles.

14. We might discuss the problem algebraically so far as the *angles* of the parts are concerned, as follows :—

In a solution of the first type there are two degrees of freedom in the position of E, and two in the position of F. Taking θ, ϕ, θ', ϕ', to denote the angles BAE, EBC, BDF, FBC respectively, it is clear that, E and F being any points in the given triangles, θ, ϕ, θ', ϕ' are aribitrary, and when chosen, determine the positions of E and F and therefore also the angles BCE, etc. Now to secure that the triangular segments shall be similar in pairs, we must for each pair equate two angles of one respectively to two angles of the other. Thus on the whole we have six equations to satisfy, by means of the four variables θ, ϕ, θ', ϕ'. The fact that there is a general solution would seem to indicate that these equations are not all independent: but I have not further considered this rather curious point, nor have I attempted to classify or discuss the great variety of ways in which the six equations can be written down, nor to determine whether the solutions I have given are the only possible ones.

Similar remarks apply to solutions of the second type.

15. The analogy of the problem here discussed to that of the dissection of figures into mutually *congruent* parts will no doubt have been noticed. Of course any dissection by which two equal triangles are made mutually congruent will serve to solve the problem of this paper for the case of two unequal triangles. But there is a special interest in the solutions given here, on account of the small number of parts necessary. This is rendered possible by the fact that there is an additional element of freedom in the problem, as compared with that which would make the ratio of each part of one triangle to the corresponding part of the other the same as that of the first whole to the second.

In Fig. 1 the angles BEC, CEA, AEB are respectively equal to $A + A'$, $B + B'$, $C + C'$; and this, as Mr John Dougall has pointed out to me, enables us to construct the point E by means of segments of circles containing these angles, described on the sides of ABC. This construction is more symmetrical. Interesting trigonometrical problems might be based on it.

Two Geometrical Transformations.

By J. A. Third, M.A., D.Sc.

The transformations discussed in the present paper are, like the isogonal and isotomic transformations, particular cases of the general birational quadratic transformation, in which points correspond to points, and lines to conics passing through three fixed points. They seem to possess some interest in connection with the Geometry of the Triangle.

1. Let K be the centre of a conic which touches the sides BC, CA, AB of the triangle of reference at X, Y, Z, and S be the point of concurrence of AX, BY, CZ. Then if x, y, z be the trilinear coordinates of S, and x', y', z' those of K, we have

$$x' : y' : z' = x(by + cz) : y(cz + ax) : z(ax + by),$$

and

$$\begin{aligned} x : y : z &= 1/a(by' + cz' - ax') \\ &: 1/b(cz' + ax' - by') \\ &: 1/c(ax' + by' - cz'). \end{aligned}$$

The following special cases may be noticed.

(i) If S be the Gergonne point, K is the incentre, with similar statements for the exterior Gergonne points.

(ii) If S be the orthocentre, K is the symmedian point.

(iii) If S be the symmedian point, K is the mid-point of the distance between the Brocard points.

(iv) If S be the centroid, K is also the centroid.

(v) If S be the isotomic conjugate of the circumcentre $\left(\frac{1}{a^2 \cos A}, \text{ etc.}\right)$, K is the nine-point centre.

(vi) If S be the isotomic conjugate of the orthocentre $\left(\frac{1}{a^2 \sec A}, \text{ etc.}\right)$, K is the circumcentre.

If K be made to describe any locus, the corresponding locus of S may be called the S-transformation of the locus of K, and the locus of K may be called the K-transformation of the locus of S.

2. The S-transformation of a straight line is a conic circumscribed to the triangle.

Demonstration. Let the locus of K be the straight line $lx+my+nz=0$; then, substituting in this equation $x(by+cz)$ for x, $y(cz+ax)$ for y, and $z(ax+by)$ for z, we obtain as the locus of S,

$$(bn+cm)yz+(cl+an)zx+(am+bl)xy=0,$$

which is a conic circumscribed to the triangle.

The following special cases call for notice.

(i) When the K-locus is the line at infinity, $ax+by+cz=0$, the S-locus is the circumconic,

$$bcyz+cazx+abxy=0,$$

i.e., the minimum circumscribed ellipse, having the centroid as centre, and well-known as the Steiner ellipse.

This theorem may be stated otherwise as follows. If the lines joining the vertices of a triangle to any point on its Steiner ellipse meet the opposite sides of the triangle in X, Y, Z, the conic which touches the sides of the triangle at X, Y, Z is a parabola.

(ii) When the K-locus is the line $x\cos A+y\cos B+z\cos C=0$, the S-locus is $$ayz+bzx+cxy=0,$$

i.e., the circumcircle of the triangle.

It may be noticed that the line $\Sigma x\cos A=0$ is perpendicular to the line joining the circumcentre and orthocentre—the Euler line. Hence using the theorem of (i), and recollecting that the fourth point of intersection of the circumcircle and the Steiner ellipse is the Steiner point of the triangle, and that the directrix of a parabola inscribed in the triangle passes through the orthocentre, we have the theorem that if the connectors of the Steiner point of a triangle with the vertices meet the opposite sides in X, Y, Z, the conic touching the sides at these points is a parabola whose directrix is the Euler line.

With regard to the Steiner point it may be remarked that it has

been defined as the point whose isogonal and isotomic conjugates lie at infinity. It now appears, from the fact that it lies on the Steiner ellipse, that its K-transformation also lies at infinity.

(iii) When the K-line passes through the symmedian point, the S-conic passes through the orthocentre (Art. 1, ii), and is, therefore, a rectangular hyperbola.

(iv) When the K-line passes through the centroid, the S-conic also passes through the centroid and touches the K-line there, for (Art. 1, iv) S and K coincide in the centroid and nowhere else.

The fact of the contact may be confirmed as follows by analysis. Since the K-line passes through the centroid $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$, its equation may be written

$$a(b\mu - c\nu)x + b(c\nu - a\lambda)y + c(a\lambda - b\mu)z = 0.$$

The equation of the S-conic is then found to be

$$\frac{b\mu - c\nu}{a} . yz + \frac{c\nu - a\lambda}{b} . zx + \frac{a\lambda - b\mu}{c} . xy = 0.$$

The condition, then, that the line should touch the conic is that

$$\Sigma(b\mu - c\nu)^4 - \Sigma 2(c\nu - a\lambda)^2(a\lambda - b\mu)^2 = 0,$$

which is the case, since

$$(b\mu - c\nu) + (c\nu - a\lambda) + (a\lambda - b\mu) = 0.$$

The theorem may be stated in the following restricted form. If K be the centre of a conic which touches the sides BC, CA, AB of a triangle at X, Y, Z respectively, and S be the point of concurrence of AX, BY, CZ, and if G is the centroid of ABC, then the conic through A, B, C, S, G touches KG at G.

(v) Suppose that the K-line passes through the vertex A. Its equation then is $my - nz = 0$, and the equation of the S-conic is, consequently, $(-bn + cm)yz - anzx + amxy = 0$.

The coordinates of the centre of this conic are found to be 0, c, b. Hence we have the theorem that when the K-line passes through a vertex of the triangle, the centre of the S-conic is the mid-point of the opposite side.

Thus, as particular cases, we have the following theorems.

(1) If the connectors of the vertices of a triangle ABC with a variable point on the circumscribed rectangular hyperbola whose centre is the mid-point of BC, meet the opposite sides in X, Y, Z, the locus of the centre of the conic which touches the sides at X, Y, Z is the symmedian through A, for, by the present section, this locus passes through A and by (iii) it also passes through the symmedian point.

(2) If Γ, Γ_1, Γ_2, Γ_3 denote the Gergonne points of ABC, Γ_1, Γ_2, Γ_3 being respectively opposite to A, B, C, the mid-point of BC is the centre of the conics through A, B, C, Γ, Γ_1 and A, B, C, Γ_2, Γ_3 for the K-transformations of those conics are obviously the internal and external bisectors respectively of the angle A.

3. The K-transformation of a straight line is a conic medioscribed to the triangle, *i.e.*, passing through the mid-points of its sides.

Demonstration. Let the locus of S be the straight line

$$lx + my + nz = 0.$$

Then, substituting in this equation $1/a(by + cz - ax)$ for x, with similar substitutions for y and z (Art. 1), we obtain as the locus of K, the conic

$$\Sigma a^2(bcl - cam - abn)x^2 + \Sigma 2b^2c^2lyz = 0.$$

The coordinates of the mid-points of the sides, viz.,

$$(0,\ c,\ b),\quad (c,\ 0,\ a),\quad (b,\ a,\ 0)$$

satisfy this equation, and therefore the conic passes through these points.

The following particular cases require notice.

(i) When the S-locus is the line at infinity, $ax + by + cz = 0$, the K-locus is

$$a^2x^2 + b^2y^2 + c^2z^2 - 2bcyz - 2cazx - 2abxy = 0.$$

As is well known, this conic is the maximum inscribed ellipse; it touches the sides of the triangle at their mid-points and has the centroid as centre.

The theorem might be stated otherwise thus. If a conic whose centre lies on the maximum inscribed ellipse of a triangle ABC touch the sides BC, CA, AB at X, Y, Z respectively, then AX, BY, CZ are parallel.

(ii) When the S-locus is the line $a^3x + b^3y + c^3z = 0$, the K-locus is

$$\Sigma a^2(a^2 - b^2 - c^2)x^2 + \Sigma 2a^2bcyz = 0,$$

i.e., $$\Sigma a\cos A \,.\, x^2 - \Sigma ayz = 0,$$

i.e., the nine-point circle of the triangle.

It may be noticed that the line $x\cos A + y\cos B + z\cos C = 0$, referred to in Art. 2, ii, and the line $a^3x + b^3y + c^3z = 0$ are parallel, and that the former is the polar of the symmedian point with respect to the maximum inscribed ellipse, while the latter is the polar of the isotomic conjugate of the orthocentre with respect to the minimum circumscribed ellipse, also that the former is related to the complementary triangle of ABC (the triangle whose vertices are the mid-points of ABC) in the same way that the latter is related to ABC.

(iii) When the S-line passes through the isotomic conjugate of the orthocentre, the K-conic passes through the circumcentre (Art. 1, vi), *i.e.*, through the orthocentre of the triangle formed by joining the mid-points of the sides. Therefore in this case the K-conic is a rectangular hyperbola.

It may be noted that the isotomic conjugate of the orthocentre is the symmedian point of the anticomplementary triangle of ABC (the triangle formed by parallels through A, B, C to the opposite sides) and is therefore related to ABC in the same way that the symmedian point is related to the complementary triangle of ABC.

(iv) When the S-line passes through the centroid, the K-conic also passes through the centroid, and touches the S-line there, for (Art. 1, iv) S and K coincide in the centroid and nowhere else.

The fact of the tangency could easily be confirmed by analysis.

Combining the statement of Art. 2, iv with the foregoing, we see that the S- and K-transformations of a straight line through the centroid touch each other and the line at the centroid. This might be stated in a variety of forms, of which the following may be given. If two conics, one circumscribed, the other medioscribed to a triangle, touch at the centroid, and if P be the centre of a conic which touches the sides of the triangle at points which connect with the opposite vertices through a point on the circumscribed conic

and if Q be the point of concurrence of the lines drawn from the vertices to meet the opposite sides at points where these sides are touched by a conic whose centre lies on the medioscribed conic, then PQ touches the circumscribed and medioscribed conics at the centroid.

4. The S-transformation of a line is a parabola, hyperbola, or ellipse, according as the line touches, cuts, or does not meet the maximum inscribed ellipse; also the K-transformation of a line is a parabola, hyperbola, or ellipse, according as the line touches, cuts, or does not meet the minimum circumscribed ellipse.

This theorem depends on another theorem which is sufficiently evident, viz., that if κ be a medioscribed conic and k a line connected with it, and if the line s and the circumscribed conic σ be the S-transformations of κ and k respectively, then if k touch κ at a point, s touches σ at the S-transformation of that point, and s cuts or does not meet σ according as k cuts or does not meet κ; and *vice versa*.

If now κ be the maximum inscribed ellipse, s is the line at infinity (Art. 3, i). Therefore, when k touches κ, σ touches the line at infinity, and is, therefore, a parabola; when k cuts κ, σ meets the line at infinity in two real points and is, therefore, a hyperbola; and when k does not meet κ, σ does not meet the line at infinity in real points, and is, therefore, an ellipse.

The second half of the proposition is proved in a precisely similar manner.

5. The isogonal transformations and the K-transformations of a system of points, lines, and circumscribed conics, are projectively related; also the isogonal transformations with respect to the triangle A'B'C', where A', B', C' are the mid-points of the sides of ABC, and the S-transformations (with respect to ABC) of a system of points, lines, and medioscribed conics, are projectively related.

These statements are involved in the more general theorem that if F be a system of lines, conics, etc., F' a system obtained by a birational quadratic transformation of F, and F'' a system obtained by another birational quadratic transformation of F, then F' and F'' can be obtained, the one from the other, by a linear transformation, and are, consequently, projective systems.

The substitutions in the case of the isogonal and K-transformations can be readily shown as follows.

Let $$\lambda x^{\pm 1} + \mu y^{\pm 1} + \nu z^{\pm 1} = 0$$

be a line or circumconic. Its isogonal transformation is

$$\lambda x^{\mp 1} + \mu y^{\mp 1} + \nu z^{\mp 1} = 0,$$

and its K-transformation

$$\lambda\{a(by + cz - ax)\}^{\mp 1} + \mu\{b(cz + ax - by)\}^{\mp 1} + \nu\{c(ax + by - cz)\}^{\mp 1} = 0.$$

Hence, in order to obtain the K-transformation from the isogonal transformation we require to substitute

$$a(by + cz - ax), \quad b(cz + ax - by), \quad c(ax + by - cz)$$

for x, y, z respectively.

The following tables, which might be indefinitely extended, may be given in illustration of the theorem of the present article.

TABLE I.

Primary elements consisting of points, lines, and circumconics.	*Derived elements, projectively related.*	
	Isogonal transformations.	*K-transformations.*
line at infinity	circumcircle	maximum inscribed ellipse
circumcircle	line at infinity	line $\Sigma x\cos A = 0$
line $\Sigma a^3 x = 0$	conic $\Sigma a^3/x = 0$	nine-point circle
minimum circumscribed ellipse	line $\Sigma x/a = 0$	line at infinity
orthocentre	circumcentre	symmedian point
centroid	symmedian point	centroid
symmedian point	centroid	mid-point of distance between Brocard points Ω and Ω'
incentre	incentre	Gergonne point
isotomic conjugate of circumcentre	isogonal conjugate of isotomic conjugate of circumcentre	nine-point centre
isotomic conjugate of orthocentre	isogonal conjugate of isotomic conjugate of orthocentre	circumcentre
etc.	etc.	etc.

TABLE II.

Primary elements consisting of points, lines, and medioscribed conics.	*Derived elements, projectively related.*	
	Isogonal transformations with respect to A′B′C′.	*S-transformations.*
line at infinity	nine-point circle	minimum circumscribed ellipse
nine-point circle	line at infinity	line $\Sigma a^2 x = 0$
line $\Sigma x \cos A$	conic $\Sigma \frac{a^2(b^2+c^2-3a^2)}{by+cz-ax} = 0$	circumcircle
maximum inscribed ellipse	line $\Sigma xa(b^2c^2-c^2a^2-a^2b^2) = 0$	line at infinity
circumcentre	nine-point centre	isotomic conjugate of orthocentre
nine-point centre	circumcentre	isotomic conjugate of circumcentre
mid-point of distance between Brocard points	point $a(b^4+c^4)$, etc., (viz., the isogonal conjugate of the isotomic conjugate of the symmedian point of A′B′C′)	symmedian point
centroid	symmedian point of A′B′C′	centroid
etc.	etc.	etc.

By means of these tables a great number of elementary theorems can be readily obtained. Thus from the second and third columns of Table I. we obtain the following.

(1) The circumcentre, the centroid, and the isogonal conjugate of the isotomic conjugate of the orthocentre are collinear (*i.e.*, the last-named point lies on the Euler line); for their correspondents in the third column, viz., the symmedian point, the mid-point of $\Omega\Omega'$, and the circumcentre lie on a straight line (the line of centres of the Tucker circles).

(2) The symmedian point, the isogonal conjugate of the isotomic conjugate of the orthocentre, and the isogonal conjugate of the isotomic conjugate of the circumcentre are collinear; for their

correspondents in the third column, viz., the centroid, circumcentre, and nine-point centre are collinear.

(3) Since the line of centres of the Tucker circles in the third column corresponds to the Euler line in the second, the S-transformation of the former line is the isogonal transformation of the latter, viz., the Jerabek hyperbola. *

(4) Since the join of the centroid and symmedian point in the third column corresponds to the line of centres of the Tucker circles in the second, the S-transformation of the former line is the isogonal transformation of the latter, viz., the Kiepert hyperbola.

From Table II. it appears that the K-transformation of the join of the isotomic conjugates of the orthocentre and circumcentre (which corresponds in the third column to the Euler line of A'B'C' in the second) is the Jerabek hyperbola of A'B'C'.

As a specimen of the theorems derivable from Table I., but not depending on the projective relation of the elements in the second and third columns, we have the following.

The isotomic conjugates of the circumcentre and orthocentre lie on a circumscribed hyperbola which passes through the centroid, and touches the Euler line at that point. This follows from the fact that the K-transformations of the three points mentioned lie on the Euler line, and from Arts. 2, iv and 4.

6. The K-transformations of a system of similar circumconics envelop a conic which has double contact with the maximum inscribed ellipse on the line $\Sigma x \cos A = 0$; and the S-transformations of a system of similar medioscribed conics envelop a conic which has double contact with the minimum circumscribed ellipse on the line $\Sigma a^3 x = 0$.

Demonstration. For convenience the elements belonging to the third column of Table I. will be called the K-projections of the corresponding elements in the second column, and the elements in the third column of Table II., the S-projections of those in the second column.

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* See Mathesis, VIII., p. 81, and Casey's Conics, p. 448.

Now the isogonal transformations of a system of similar circumconics are, as is well known, tangents to a circle, U, concentric with the circumcircle, and consequently having (imaginary) double contact with the circumcircle on the line at infinity. Therefore, by the preceding article, the K-transformations of the same system are tangents to the conic which is the K-projection of U, *i.e.*, to a conic having double contact with the K-projection of the circumcircle (the maximum inscribed ellipse) on the K-projection of the line at infinity (the line $\Sigma x \cos A = 0$). Thus the first part of the proposition is proved.

Similarly the isogonal transformations, with respect to the triangle A'B'C', of a system of similar medioscribed conics, are tangents to a circle, V, concentric with the nine-point circle, and consequently having (imaginary) double contact with the nine-point circle on the line at infinity. Therefore the S-transformations of the system are tangents to the conic which is the S-projection of V, *i.e.*, to a conic having double contact with the S-projection of the nine-point circle (the minimum circumscribed ellipse) on the S-projection of the line at infinity (the line $\Sigma a^2 x = 0$). Thus the proposition is completely proved.

The converse may be stated. The S-transformations of tangents to a conic having double contact with the maximum inscribed ellipse on the line $\Sigma x \cos A = 0$, are circumconics having the same eccentricity; and the K-transformations of tangents to a conic having double contact with the minimum circumscribed ellipse on the line $\Sigma a^2 x = 0$, are medioscribed conics having the same eccentricity.

The special cases when the transformations are circles, equilateral hyperbolas and parabolas have already been dealt with in Arts. 2 ii, 3 ii, 2 iii, 3 iii and 4.

7. As another special case of the general theorem given in Art. 5, we have the following theorems which indicate the remarkable connection existing between the isotomic transformation on the one hand (where $1/a^2x$, $1/b^2y$, $1/c^2z$ are substituted for x, y, z in the equations of the original elements) and the S- and K-transformations on the other.

(i) If S and K be two points (or loci) connected with a triangle

ABC in such a manner that the first is the S-transformation of the second, and consequently the second the K-transformation of the first, the isotomic transformation, S′, of S bears to ABC the same relation that K bears to the complementary triangle A′B′C′.

Demonstration. Let the coordinates of K be x, y, z. Then those of S are

$$1/a(by+cz-ax), \quad 1/b(cz+ax-by), \quad 1/c(ax+by-cz)$$

and consequently those of S′

$$(by+cz-ax)/a, \quad (cz+ax-by)/b, \quad (ax+by-cz)/c.$$

The coordinates of K referred to the triangle A′B′C′ are

$$p-x, \quad q-y, \quad r-z,$$

where p, q, r are the altitudes of A′B′C′,

$$i.e., \quad \frac{\Delta}{a}-x, \quad \frac{\Delta}{b}-y, \quad \frac{\Delta}{c}-z,$$

where Δ is the area of ABC,

$$i.e., \quad (by+cz-ax)/2a, \quad (cz+ax-by)/2b, \quad (ax+by-cz)/2c.$$

Therefore K is related to the triangle A′B′C′ in the same way that S′ is related to the triangle ABC.

Another statement of the same fact is that K is related to ABC in the same way that S′ is related to the anticomplementary triangle of ABC.

The following are obvious corollaries.

(*a*) The join of K to any of the vertices of ABC (or A′B′C′) is parallel to the join of S′ to the corresponding vertex of the anticomplementary triangle of ABC (or ABC itself). Also if K_1, K_2 be two positions of K, and S'_1, S'_2 the corresponding positions of S′, then K_1K_2 and $S'_1S'_2$ are parallel and have opposite directions, and $K_1K_2 = \frac{1}{2}S'_1S'_2$.

(*b*) The centroid is a point of trisection of KS′.

(*c*) The loci of K and S′ are homothetic, the centroid being the homothetic centre.

An interesting particular case of (i) arises when K is at infinity, *i.e.*, when the inconic of which K is the centre is a parabola. We then obtain, with the help of (*a*) above, the following theorem which I cannot recollect having seen stated, viz., that if ABC is the triangle formed by three tangents to a parabola, and if X be the point of contact on BC, and X′ be the point where the diameter through A meets BC, then $BX = CX'$. From this can be derived another theorem relating to the parabola, also new to me, viz., that if ABC be the triangle formed by three tangents to a parabola and if X be the point of contact on BC, then if $AB = AC$, A, X and the focus are collinear, and conversely.

When K is an incentre or excentre, we have, as a particular case, the well-known theorem that the isotomic conjugate of the corresponding Gergonne point (Nagel point) is the incentre or excentre as the case may be of the anticomplementary triangle.

It may be noted also that since any line passing through the centroid is related in the same way to both ABC and A′B′C′, the isotomic transformation of such a line is the same as its S-transformation.

(ii) It follows from (i) above that S bears to ABC the same relation that the isotomic conjugate of K, with respect to A′B′C′, bears to A′B′C′.

Combining (i) and (ii) we have the following remarkable theorem, viz., that if P be an element (point or locus) connected with the triangle ABC, and P′ be the corresponding element connected with the complementary triangle A′B′C′, then the K-transformation of P is related to A′B′C′ in the same way that the S-transformation of P′ is related to ABC.

The following table illustrates the correspondence between the isotomic and the K-transformations. A similar table could easily be constructed showing the correspondence between the isotomic transformation, with respect to A′B′C′, and the S-transformation.

TABLE III.

Primary elements consisting of points, lines, and circumconics.	*Derived elements.*	
	Isotomic transformations.	*K-transformations.*
line at infinity	minimum circumscribed ellipse	maximum inscribed ellipse
minimum circumscribed ellipse	line at infinity	line at infinity
line $\Sigma a^2 x = 0$	circumcircle	nine-point circle
circumcircle	line $\Sigma a^2 x = 0$	line $\Sigma x \cos A = 0$
centroid	centroid	centroid
orthocentre	isotomic conjugate of orthocentre (symmedian point of anticomplementary triangle)	symmedian point
isotomic conjugate of orthocentre	orthocentre	circumcentre (orthocentre of A'B'C')
symmedian point	isotomic conjugate of symmedian point (midpoint of the distance between the Brocard points of the anticomplementary triangle)	mid-point of the distance between the Brocard points
isotomic conjugate of circumcentre	circumcentre	nine-point centre
Gergonne point	isotomic conjugate of Gergonne point (Nagel point)	corresponding in- or ex-centre
etc.	etc.	etc.

It may be noticed that points in the third column are derived from points in the second by the substitution $(by + cz)/a$ for x, etc., and points in the second from points in the third by the substitution $(by + cz - ax)/a$ for x, etc. These substitutions have already been discussed by Mons. E. Lemoine,* who, however, makes no mention of the isotomic or K-transformation in connection with them.

* E. Lemoine: Notes de Géométrie, Deux modes de génération de quelques points remarquables, Association française, Congrès de Besançon, 1893.

8. The following generalisations seem to be worth stating.

Let a variable conic touch the sides BC, CA, AB of a triangle at X, Y, Z, and let S be the point of concurrence of AX, BY, CZ. Let a fixed transversal meet BC, CA, AB in L, M, N, and let L′, M′, N′ be the harmonic conjugates of these points with respect to the ends of the sides on which they lie. Let P be the pole of LMN with respect to the conic.

Then, by projection, we obtain the following from theorems already established.

(i) If the locus of P is a straight line, the locus of S is a circumconic (general theorem of Art. 2).

(ii) If the locus of P is LMN, *i.e.*, if the conic touches LMN as well as the sides of the triangle, the locus of S is the circumconic which touches AL, BM, CN (Art. 2, i).

(iii) If the locus of S is a straight line, the locus of P is a conic through L′, M′, N′ (general theorem of Art. 3).

(iv) If the locus of S is LMN, the locus of P is the conic which touches the sides of ABC at L′, M′, N′ (Art. 3, i).

From each of the foregoing a correlative theorem could be obtained by reciprocation.

Second Meeting, 8th December 1899.

R. F. MUIRHEAD, Esq., M.A., B.Sc., President, in the Chair.

On the Evaluation of a certain Determinant.

By LAWRENCE CRAWFORD, M.A., D.Sc.

1. In a paper by Mr Arthur Berry, M.A., in the *Proceedings of the Cambridge Philosophical Society*, Volume X. Pt. I., "On the Evaluation of a certain Determinant which occurs in the mathematical theory of statistics and in that of elliptic geometry of any number of dimensions," a remark is made that in the case of $n=3$ this determinant was readily evaluated by me by means of the formulæ of spherical trigonometry. I have thought that it might be of interest to show this evaluation, but I shall merely state the determinant at once of order 3, and leave the reader to refer to the paper quoted for the general determinant.

2. The question then is, to evaluate the determinant

$$\frac{\partial(\xi_{12},\ \xi_{13},\ \xi_{23})}{\partial(r_{12},\ r_{13},\ r_{23})},$$

where $\xi_{12}=R_{12}/\sqrt{R_{11}R_{22}}$, &c., where R_{pq} is the minor (with proper sign) of the element in the pth row and qth column of R, where

$$R=\begin{vmatrix} 1 & r_{12} & r_{13} \\ r_{12} & 1 & r_{23} \\ r_{13} & r_{23} & 1 \end{vmatrix}.$$

3. We have $R_{11}=1-r_{23}^2$, $R_{22}=1-r_{13}^2$, $R_{33}=1-r_{12}^2$;
$R_{12}=r_{13}r_{23}-r_{12}$, $R_{13}=r_{12}r_{23}-r_{13}$, $R_{23}=r_{12}r_{13}-r_{23}$.

Put then $r_{23}=\cos a$, $r_{13}=\cos b$, $r_{12}=\cos c$, where a, b, c are the sides of a spherical triangle, then $R_{11}=\sin^2 a$, $R_{22}=\sin^2 b$, $R_{33}=\sin^2 c$,

and $R_{12} = \cos b \cos a - \cos c = -\sin a \sin b \cos C$, $R_{13} = -\sin a \sin c \cos B$,
and $R_{23} = -\sin b \sin c \cos A$,

$$\therefore \quad \xi_{12} = -\cos C, \quad \xi_{13} = -\cos B, \quad \xi_{22} = -\cos A.$$

$\therefore$ we wish J, where $J = -\dfrac{\partial(\cos C, \cos B, \cos A)}{\partial(\cos c, \cos b, \cos a)}$

$$= -\frac{\partial(\cos A, \cos B, \cos C)}{\partial(\cos a, \cos b, \cos c)};$$

$$\therefore \quad J \times \frac{\partial(\cos a, \cos b, \cos c)}{\partial(a, b, c)} = -\frac{\partial(\cos A, \cos B, \cos C)}{\partial(a, b, c)}.$$

In this a, b, c are independent, A, B, C functions of a, b, c.

$$\cos A = (\cos a - \cos b \cos c)/\sin b \sin c,$$

$$\therefore \quad \frac{\partial(\cos A)}{\partial a} = -\sin a/\sin b \sin c,$$

and $$\frac{\partial(\cos A)}{\partial b} = \{\sin^2 b \cos c - (\cos a - \cos b \cos c)\cos b\}/\sin^2 b \sin c,$$

$$= (\cos c - \cos a \cos b)/\sin^2 b \sin c,$$

$$= \sin a \cos C/\sin b \sin c,$$

and $$\frac{\partial(\cos A)}{\partial c} = \sin a \cos B/\sin b \sin c.$$

$$\therefore J \times \frac{\partial(\cos a, \cos b, \cos c)}{\partial(a, b, c)} = \frac{-1}{\sin^2 a \sin^2 b \sin^2 c} \begin{vmatrix} -\sin a & \sin a \cos C & \sin a \cos B \\ \sin b \cos C & -\sin b & \sin b \cos A \\ \sin c \cos B & \sin c \cos A & -\sin c \end{vmatrix}$$

$$= \frac{-1}{\sin a \sin b \sin c} \begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$$

$$= -(-1 + 2\cos A \cos B \cos C + \cos^2 A + \cos^2 B + \cos^2 C)/\sin a \sin b \sin c$$

$$= (1 + 2\cos a' \cos b' \cos c' - \cos^2 a' - \cos^2 b' - \cos^2 c')/\sin a \sin b \sin c$$

(using the polar triangle)

$$= \sin^2 A' \sin^2 b' \sin^2 c'/\sin a \sin b \sin c$$

$$= \sin^2 a \sin^2 B \sin^2 C/\sin a \sin b \sin c$$

$$= \sin^2 A \sin B \sin C/\sin a.$$

$$\text{Now } \frac{\partial(\cos a, \cos b, \cos c)}{\partial(a, b, c)} = \begin{vmatrix} -\sin a & 0 & 0 \\ 0 & -\sin b & 0 \\ 0 & 0 & -\sin c \end{vmatrix} = -\sin a \sin b \sin c;$$

$$\therefore \quad J = -\sin^2 A \sin B \sin C / \sin^2 a \sin b \sin c$$

$$= -(\sin A / \sin a)^4.$$

$$\text{Also } R = \quad 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c$$

$$= \quad \sin^2 A \sin^2 b \sin^2 c\ ;$$

$$\therefore \quad J = -(1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c)^2 / \sin^4 a \sin^4 b \sin^4 c\ ;$$

$$\therefore \quad J = -R^2 / R_{11}^2 R_{22}^2 R_{33}^2.$$

A special case of the dissection of any two triangles into mutually similar pairs of triangles.

By Alex. D. Russell, B.Sc.

This note is intended to be supplementary to the paper, by Mr Muirhead, on "The dissection of any two triangles into mutually similar pairs of triangles." The constructions given there, for the general case of this problem, yield no real solution if one angle of one triangle be greater than the sum of any two angles of the other triangle. For this particular case, the following constructions supply the necessary requirements; the first leads to a division of the triangles into three parts, the second to a division into four parts.

Let ABC and DEF be the two triangles and let the angle BAC be greater than the sum of any two angles of the triangle DEF.

First Construction (Fig. 5).

At A make $\angle BAG = \angle DFE$ and $\angle CAH = \angle DEF$,
and at D make $\angle FDK = \angle ABG$ and $\angle EDL = \angle ACH$.

Then $\angle GAH = \angle KDL$ (Euc. I. 32),

and $\angle AHG = \angle CAH + \angle ACH = \angle DEL + \angle EDL = \angle DLK$.

By comparing the angles it is evident that the following pairs of triangles are equiangular and therefore similar:—

ABG and FDK, ACH and EDL, AHG and DLK.

The first two pairs are directly similar, the third pair perversely. By a slightly different arrangement of the angles there may be only one pair directly similar and the other two pairs perversely.

Second Construction (Fig. 6).

$\angle BAC > \angle DEF + \angle DFE$, $\therefore$ $\angle EDF > \angle ACB$.

At D make $\angle FDL = \angle ACB$; at A make $\angle BAK = \angle EDL$; then since $\angle KAC < \angle DEL + \angle DFL$, AG may be drawn in such a

position that $\angle KAG < \angle DEL$ and $\angle GAC < \angle DFL$; at E make $\angle LEM = \angle KAG$; at B make $\angle ABH = \angle DEM$, and at F make $\angle DFN = \angle GAC$. Let the lines meet as in the figure.

Then $\angle HBG = \angle LFN$ (I. 32);

$\angle AKH = \angle BAK + \angle ABK = \angle EDM + \angle DEM = \angle EML$;

$\angle HGB = \angle GAC + \angle GCA = \angle NFD + \angle NDF = \angle LNF$;

The following pairs of triangles are then obviously similar:—

ABK and DEM, BGH and FNL, AKH and EML, CAG and DFN.

The two first-mentioned pairs are directly similar, the remaining two pairs perversely.

Elementary Proof of the Potential Theorems regarding Uniform Spherical Shells.

By Dr Peddie.

Let P be the point at which the potential has to be found (Fig. 7). Let the uniform surface density be σ, and take two radii PQ, PQ′ differing in length by a small quantity Q′K. Let QR be drawn perpendicular to BD, the diameter through P, and let CL be drawn perpendicular to PQ.

If we consider an elementary zone of the surface surrounding the diameter CP, we get

$$2\pi\sigma \mathrm{QR} \,.\, \mathrm{QQ}' = m,$$

where m is the mass of the zone.

Also, by similar triangles, we have

$$\mathrm{Q}'\mathrm{K} \,.\, \mathrm{CQ} = \mathrm{QQ}' \,.\, \mathrm{CL},$$

and

$$\mathrm{CL} \,.\, \mathrm{PQ} = \mathrm{QR} \,.\, \mathrm{CP}.$$

Therefore

$$m = 2\pi\sigma \mathrm{Q}'\mathrm{K} \,.\, \mathrm{CQ} \,.\, \mathrm{PQ}/\mathrm{CP}.$$

Thus the mass per unit difference of radii is $2\pi a\sigma \mathrm{PQ}/\mathrm{CP}$, where $a = \mathrm{CQ}$, the radius of the shell; and the potential, at P, of a zone corresponding to unit difference of radii is

$$2\pi a\sigma/\mathrm{CP},$$

which is constant.

Therefore, summing over the total difference of radii, the potential, at P, of the whole shell is

$$2\pi a\sigma \frac{\mathrm{PD} - \mathrm{PB}}{\mathrm{CP}}.$$

At an inside point this becomes $4\pi a\sigma = \mathrm{M}/a$, where M is the whole mass of the shell. At an outside point it becomes

$$4\pi a^2\sigma/\mathrm{CP} = \mathrm{M}/\mathrm{CP}.$$

[*Note, added Jan. 1900.*—Mr Muirhead will communicate to the Society a neat modification of the above proof, which avoids any assumption of the expression for the surface of a sphere. It is interesting to note that the above proof can be used to give a *physical* determination of the value of the surface. For, since, at *any* inside point, the potential has the value $4\pi a\sigma$, at the centre the potential is $4\pi a^2\sigma/a$. Hence, from the definition of potential, the surface is $4\pi a^2$.—W. P.]

Third Meeting, 12*th January*, 1900.

R. F. MUIRHEAD, Esq., M.A., B.Sc., President, in the Chair.

On the Dissipation of Energy in Vibrating Matter.

By Dr PEDDIE.

The Society adopted the following motion, proposed by Dr MACKAY and seconded by Mr ALISON:—

> "The Edinburgh Mathematical Society resolves that Professor GIBSON's paper on Proportion, in the final form which he has now given to it, be printed in its *Proceedings*, and recommends it to Mathematical Teachers as a suitable and sufficient substitute for Euclid's Fifth Book."

Professor GIBSON's paper is printed as an Appendix to this Session's *Proceedings*.

Fourth Meeting, Friday, 9th February, 1900.

R. F. MUIRHEAD, Esq., M.A., B.Sc., President, in the Chair.

Remark on Dr Peddie's Proof of the Potential Theorems regarding Uniform Spherical Shells.

By R. F. MUIRHEAD, M.A., B.Sc.

On reading Dr Peddie's paper, the following modification of the proof, which avoids summation, occurred to me:—

If in Figure 7 we take a point S on the circle BQD such that $PQ + PS = 2a$, where a is the radius, and a corresponding point S' such that $PQ' + PS' = 2a$, then it is clear by Dr Peddie's construction that the potential at P due to the zone of the spherical surface lying between planes through Q and Q' perpendicular to BD is given by $2\pi\sigma(PQ' - PQ) \cdot a/CP$, and is therefore equal to that due to the corresponding zone between S and S', since

$$PQ' - PQ = PS - PS'.$$

The potentials at P due to these zones being respectively $\frac{m}{PQ}$ and $\frac{m'}{PS}$, where m and m' are the masses of these zones, and these potentials being equal, their sum is

$$\frac{2(m+m')}{PQ+PS} = \frac{2(m+m')}{2a} = \frac{m+m'}{a}.$$

Thus the potential due to these parts of the surface is the same as if they were placed at distance a from P. But since the whole spherical surface is divisible into such corresponding pairs of zones, the potential at P due to the whole surface is the same as if its mass were *all* at distance a from P, *i.e.*, the same as when P is at C.

The foregoing applies to the case when P is an internal point, but the modification for the case of P external is easily made.

A general mechanical description of the Conic Sections.

By ALEX. MORRISON, M.A., B.Sc.

Fifth Meeting, March 9th, 1900.

R. F. MUIRHEAD, Esq., M.A., B.Sc., President, in the Chair.

The determination of Green's Function by means of Cylindrical or Spherical Harmonics.

By JOHN DOUGALL, M.A.

INTRODUCTION.

In the following paper, the determination of Green's function, for spaces bounded by surfaces of the cylindrical and spherical polar systems, is effected by what is believed to be a novel process, in which are utilised the properties of cylindrical and spherical harmonics, regarded as functions of their parameters.

The functions considered are the cylindrical harmonic

$$(Ae^{\kappa z} + Be^{-\kappa z})B_m(\kappa\rho)\quad (C\cos m\phi + D\sin m\phi)$$

and the spherical harmonic

$$(Ar^n + Br^{-n-1})S_n^m(\cos\theta)(C\cos m\phi + D\sin m\phi)\,;$$

ρ, z, ϕ and r, θ, ϕ being the cylindrical and spherical coordinates of a point in space. Here, as is known, $B_m(\kappa\rho)$ or $S_n^m(\cos\theta)$ can be so chosen that the function in which it occurs as a factor is a potential function, that is, satisfies Laplace's equation, $\nabla^2 V = 0$.

Taking $S_n^m(\cos\theta)$ for example, S must satisfy a certain well-known differential equation of the second order, in which $\mu \equiv \cos\theta$ is the independent variable. In this order of procedure the parameters n, m, the *degree* and *rank* of the harmonic, occupy the place of constants, and the nature of the functionality of S, as depending on n, m, does not come into question. It is, however, possible to define two solutions, in general independent, of the equation for S, which are, for a given μ, continuous and indeed holomorphic functions of the complex variables m and n. For most applications, it is sufficient to regard one only of the parameters as variable; the other, along with the geometrical variable μ, is for the moment constant. Then, supposing for example that μ, m are given, the properties of holomorphic function of n we have to consider are chiefly those relating to the distribution of the zeros, and the nature

of the singularity at infinity. The importance of these questions has in special cases long been recognised. Thus, for instance, Laplace's investigation of an approximate formula for the Legendre's coefficient of high order, clearly essentially belongs to the theory of the zonal harmonic as a function of its degree.

Of the Bessel functions, a similar view may be taken. We define two solutions of Bessel's equation, $J_m(\kappa\rho)$, $G_m(\kappa\rho)$; as functions of the *factor* κ, the properties of these solutions are well known; as functions of the *rank* m, they have received little consideration, although they possess many simple and elegant properties, which may be usefully applied in various problems of physical mathematics.

The first part of the paper is occupied with the definitions of the functions used, and, in the case particularly of the spherical harmonics, with a short sketch of their leading properties. The methods here are perhaps to some extent novel, but for general conceptions I am greatly indebted to a valuable memoir by Dr Hobson in the *Phil. Trans.*, 1896.

The second part of the paper is devoted to the determination of Green's function. The problem is, to determine a potential function which shall be zero at the boundary of a given space, and discontinuous at only one point within the space, at which point, or pole, it becomes infinite as $1/r$, in the usual mode of expression; that is, the difference between Green's function and the reciprocal of the distance from the pole must tend to a definite limit as the variable point approaches the pole. The method of solving the problem may be described as *direct;* the function $1/r$ is taken as basis, and Green's function found by adding to this a function, continuous throughout the space, and neutralising, or balancing, $1/r$ at the boundary. For the application of this direct method we require first of all a representation of the function $1/r$ in terms of the appropriate harmonic functions. For the Bessels and spherical harmonics, two such representations are already well known, and are immediately available for the purpose of neutralising $1/r$ at cylindrical, or parallel plane boundaries, in the case of the Bessels; at conical, or spherical boundaries in the case of the spherical harmonics. A third representation, the natural complement of the other two, is here given; this involves harmonics of pure imaginary rank, and serves the purpose of neutralising $1/r$ at a boundary consisting of two axial planes.

When in this way the Green's function has been found for, say, a space bounded by two spheres, the difficulty of this method hitherto has been to neutralise this Green's function at additional boundaries, without disturbing the balance already attained at the first boundary. This difficulty is here overcome very simply by means of transformations depending on the use of Cauchy's fundamental integration theorem. Thus, just as we find three forms for the fundamental function $1/r$, so also three forms are found for the Green's function for a space bounded by, say, two spheres. One of these forms can be immediately applied to neutralise the Green's function at one or two additional conical boundaries; the other of the two forms may be similarly used when the additional boundary consists of two axial planes. For each of these new Green's functions, three forms are likewise obtained, and so the process can be continued. In all cases the function is expressed by means of a double series of harmonics, or the integral of a series, or a double integral.

The advantage of possessing alternative forms for the functions does not cease when the analytical transformations are completed; each form has, in fact, its own particular region of very rapid convergence, and hence, from the purely arithmetical point of view, it is useful to have all.

A totally different and, from many points of view, extremely beautiful method of dealing with the problem, has been given by Stokes, who takes the case of a finite space bounded by the six faces of a rectangular solid. This method consists essentially in assuming for the function required an expansion in the form of a triple series of functions, which are not potentials, but each of which vanishes at every part of the boundary. The coefficients are determined by differentiation and integration. The triple series can be reduced to a double series in three different ways. These correspond to the three forms alluded to above.

There is nothing to prevent the application of this method to all the cases given below. (For an example, see a paper by Mr H. M. Macdonald, L.M.S., vol. 26, "The electrical distribution on a conductor, etc.") The advantage of the method of this paper is that it is independent of any theorem for the expansion of an arbitrary function.

The Bessel Functions.

1. The definition of the Bessel Function of the first kind is firmly established, having come down, indeed, from Bessel himself, viz. :—

$$J_m(x) = \sum_s (-)^s \frac{x^{m+2s}}{2^{m+2s}\Pi(s)\Pi(m+s)}.$$

(Throughout the paper $\sum_s f(s)$ means, in the absence of a statement to the contrary, $f(0)+f(1)+f(2)+\ldots\ldots$, and $\sum'_s f(s)$ means, $\frac{1}{2}f(0)+f(1)+f(2)+\ldots\ldots$)

For the Bessel Function of the second kind, various definitions and symbols are, unfortunately, prevalent. The suggestion of Gray and Matthews is here adopted, so that we define

$$G_m(x) = \frac{\pi}{2\sin m\pi}\left\{J_{-m}(x) - e^{-im\pi}J_m(x)\right\}.$$

If m be a given arbitrary constant, J and G are functions of x, in general multiform. They become perfectly defined when the phase of x is restricted to a range of 2π; this range we usually take to be from $-\frac{\pi}{2}$ to $+\frac{3\pi}{2}$. The form of the functions in the vicinity of the singular point $x=0$ is at once evident from the definitions, but the case of m an integer is exceptional. In this case $J_m(x)$ is uniform and $J_{-m}x = (-)^m J_m x$; the definition of $G_m x$ becomes illusory. For this special case, we define G_m as the limit of the function which defines it in the general case. The singularity at $x=0$ is then logarithmic and $G_m(x) = -J_m(x)\log x +$ a uniform function of x.

For both J and G, $x=\infty$ is an essential singularity. All the information necessary as to the form of the functions in the vicinity of $x=\infty$, is obtainable from the very important semi-convergent expansion of G. We have when $x>0$

$$G_m(x) = e^{-\frac{m\pi i}{2}} e^{i(x+\frac{\pi}{4})}\sqrt{\frac{\pi}{2x}}\sum_{p=0}^{p=r-1}\frac{\Pi(p-m-\frac{1}{2})\Pi(p+m-\frac{1}{2})}{\Pi(-m-\frac{1}{2})\Pi(m-\frac{1}{2})\Pi p}(2ix)^{-p} + R,$$

where if $r>m-\frac{1}{2}$, and m be real, R is less in absolute value than the next, or $(r+1)^{\text{th}}$ term of the series.

The formula holds if $-\frac{\pi}{2} <$ phase of $x < \frac{3\pi}{2}$.

The corresponding formula for $J_m(x)$ is obtained from the equation $\pi i J_m(x) = G_m(x) - e^{im\pi} G_m(xe^{i\pi})$, which follows at once from the definition of G. If the real part of x is negative, we replace this by

$$\pi i J_m(x) = e^{im\pi} G_m(xe^{-i\pi}) - e^{2im\pi} G_m(x).$$

But whether m be real or not, we have for the limiting forms of the functions for x infinite,

$$G_m(x) = e^{-\frac{m\pi i}{2}} e^{i(x+\frac{\pi}{4})} \sqrt{\frac{\pi}{2x}};$$

$$J_m(x) = \sqrt{\frac{2}{\pi x}} \cos\left\{x - (m+\tfrac{1}{2})\frac{\pi}{2}\right\}, \text{ if real part of } x \text{ is positive;}$$

$$= e^{m\pi i} J_m(xe^{-i\pi}), \text{ if real part of } x \text{ is negative.}$$

At infinity in the upper part of the plane, therefore,

$J_m(x)$ is infinite, $G_m(x)$ vanishes, and $G_m(ax)J_m(bx)$,

where a, b are real positives, vanishes if $a > b$.

2. As regards the zeros of J and G as functions of x, it is sufficient to state here the following well-known theorems:—

(*a*) When m is real and positive, the zeros of $J_m(x)$ are all real and simple, but
$G_m(x)$ has no zeros for which the imaginary part of x is positive or zero.

(*b*) $J_m(ax)G_m(bx) - J_m(bx)G_m(ax)$, where a, b are real and positive, is a uniform, even function of x, whose zeros are all real and simple.

3. Passing to the consideration of J, G as functions of their rank m, we note first that the functions are holomorphic, and therefore, for instance, expansible in series of ascending powers of m convergent over the whole plane; the x in $J_m(x)$, $G_m(x)$ is supposed to have any constant value, but the particular value $x = 0$ is excluded from consideration.

The forms of the functions for an m of very large modulus are given at once by the defining expansions.

Thus, for a very large m, we have approximately

$$J_m(x) = \frac{x^m}{2^m \Pi m}.$$

Also $$\Pi m = \sqrt{2\pi m}\left(\frac{m}{e}\right)^m = \sqrt{2\pi m}\, e^{m\log m - m};$$

this formula holds if the phase of m lie between $-\pi$, $+\pi$. For the succeeding applications the only case of importance is that in which x is a positive pure imaginary, ia say, where a is real and positive.

We then have modulus of

$$e^{-\frac{m\pi i}{2}} J_m(ia) = \frac{1}{\sqrt{2\pi M}}\left(\frac{ae}{2M}\right)^{M\cos\alpha} e^{M\sin\alpha},$$

$$\text{where } m = Me^{i\alpha}; \quad -\pi < \alpha < \pi;$$

$$\text{or modulus } J_m(ia) = \frac{1}{\sqrt{2\pi M}}\left(\frac{ae}{2M}\right)^{M\cos\alpha} e^{-M\left(\frac{\pi}{2}-\alpha\right)\sin\alpha}.$$

Again, we have

$$e^{\frac{m\pi i}{2}} G_m(ib) = \frac{\pi}{2\sin m\pi}\left\{e^{\frac{m\pi i}{2}} J_{-m}(ib) - e^{-\frac{m\pi i}{2}} J_m(ib)\right\}$$

$\therefore$ $e^{\frac{m\pi i}{2}} G_m(ib)$ is an *even* function of m. Taking b to be real and positive, this function is real for m real, and therefore for m a pure imaginary; further, for the four values of m, $\pm p \pm iq$, where p, q are real, the function has the same modulus.

Consider then an m in the first quadrant.

We have
$$e^{\frac{m\pi i}{2}} G_m(ib) = \frac{\pi}{2\sin m\pi} e^{\frac{im\pi}{2}} J_{-m} ib$$
$$= \frac{\Pi m}{2m}\left(\frac{b}{2}\right)^{-m} \text{ approximately.}$$

This holds except for m a pure imaginary, and therefore, except for such values of m, $G_m(ib)$ is infinite with m. For a pure imaginary m, say $m = is$, s real and positive, we have

$$e^{\frac{m\pi i}{2}} G_m(ib) = \frac{1}{2m}\left\{\Pi(m)\left(\frac{b}{2}\right)^{-m} - \Pi(-m)\left(\frac{b}{2}\right)^{m}\right\}$$

and therefore

$$G_{is}(ib) = \sqrt{\frac{2\pi}{s}}\sin\left\{\frac{\pi}{4} + s\log s - s - \log\frac{b}{2}\right\}.$$

Lastly, the function $J_m(\lambda a)G_m(\lambda b) - J_m(\lambda b)G_m(\lambda a)$,

being equal to $\frac{\pi}{2\sin m\pi}\{J_m(\lambda a)J_{-m}(\lambda b) - J_m(\lambda b)J_{-m}(\lambda a)\}$,

is a uniform even function of m, just as it is of λ.

When m is large, this function

$$= \frac{1}{2m}\left\{\left(\frac{a}{b}\right)^m - \left(\frac{b}{a}\right)^m\right\}$$

$$= \frac{1}{m}\sinh(m\log a/b).$$

It is interesting to compare this formula with its analogue for m fixed, λ very great, namely,

$$\frac{1}{\lambda\sqrt{ab}}\sin\lambda(a-b).$$

4. We require next a few theorems relative to the zeros of the functions of m, for the proof of which the following definite integrals are convenient.

Let u, v be any Bessel Functions of ranks m, n, and factors κ, λ respectively;

so that
$$\rho\frac{d^2u}{d\rho^2} + \frac{du}{d\rho} + \left(\kappa^2\rho - \frac{m^2}{\rho}\right)u = 0$$

$$\rho\frac{d^2v}{d\rho^2} + \frac{dv}{d\rho} + \left(\lambda^2\rho - \frac{n^2}{\rho}\right)v = 0.$$

Multiply these equations by v, u, respectively, subtract, and integrate from a to b.

Thus $$\int_a^b \left\{(\kappa^2 - \lambda^2)\rho + \frac{n^2 - m^2}{\rho}\right\}uv\,d\rho = \left|\rho\left(u\frac{dv}{d\rho} - v\frac{du}{d\rho}\right)\right|_a^b.$$

The consequences which follow from taking $m^2 = n^2$ are familiar; we obtain results of precisely analogous significance for the theory of the functions of m, by taking $\kappa^2 = \lambda^2$.

(i) Take $\left.\begin{matrix} u = G_m\lambda\rho \\ v = G_n\lambda\rho \end{matrix}\right\}$ λ a positive pure imaginary.

Then $\int_1^\infty G_m(\lambda\rho)G_n(\lambda\rho)\frac{d\rho}{\rho} = \frac{\lambda}{n^2-m^2}\left\{G_n\lambda \cdot \frac{d}{d\lambda}G_m\lambda - G_m\lambda\frac{d}{d\lambda}G_n\lambda\right\}$

$$\int_1^\infty (G_m\lambda\rho)^2\frac{d\rho}{\rho} = \frac{\lambda}{2m}\left\{\frac{dG\lambda}{dm}\cdot\frac{dG\lambda}{d\lambda} - G\lambda\frac{d^2G\lambda}{dmd\lambda}\right\}.$$

so that if $G_m(\lambda)=0$,

$$\int_1^\infty G_m(\lambda\rho)^2\frac{d\rho}{\rho} = \frac{\lambda}{2m}\,\frac{dG\lambda}{dm}\cdot\frac{dG\lambda}{d\lambda}.$$

(ii) Similarly if the real part of $m+n$ be positive,

$$\int_0^1 J_m(\lambda\rho)J_n(\lambda\rho)\frac{d\rho}{\rho} = \frac{\lambda}{m^2-n^2}\left\{J_n(\lambda)\frac{d}{d\lambda}J_m(\lambda) - J_m(\lambda)\frac{d}{d\lambda}J_n(\lambda)\right\}.$$

$$\int_0^1 (J_m\lambda\rho)^2\frac{d\rho}{\rho} = \frac{\lambda}{2m}\left\{J\lambda\frac{d^2J\lambda}{dmd\lambda} - \frac{dJ\lambda}{d\lambda}\,\frac{dJ\lambda}{dm}\right\}.$$

(iii) Take $u = J_m\lambda\rho \,.\, G_m\lambda b - J_m\lambda b \,.\, G_m\lambda\rho$

$$v = J_n\lambda\rho \,.\, G_n\lambda b - J_n\lambda b \,.\, G_n\lambda\rho$$

Then $\int_a^b uv\frac{d\rho}{\rho} = \frac{a}{n^2-m^2}\left(v\frac{du}{d\rho} - u\frac{dv}{d\rho}\right)_{\rho=a}.$

$$\int_a^b u^2\frac{d\rho}{\rho} = \frac{a}{2m}\left\{\frac{du}{dm}\,\frac{du}{d\rho} - u\frac{d^2u}{dmd\rho}\right\}_{\rho=a}.$$

5. We now prove the few theorems we require relative to the zeros of the functions of m.

In the following λ is a positive pure imaginary; a and b real and positive.

(a) $J_m(\lambda)$ has no zeros with real part positive or nil.

For $i^{-m}J_m(\lambda)$ is real for m real; therefore for $m=p\pm iq$, the values of $i^{-m}J_m\lambda\rho$ are conjugate complexes;

therefore $\int_0^1 J_{m1}(\lambda\rho)J_{m2}(\lambda\rho)\frac{d\rho}{\rho}$ cannot vanish,

where $m_1=p+iq$, $m_2=p-iq$; therefore by the first result of 4 (ii) $J_{m1}(\lambda)$, $J_{m2}(\lambda)$ cannot both vanish, and hence neither can vanish; that is, $J_m\lambda$ has no *complex* zeros.

It has no *real* zeros; otherwise we should have a potential

function, say $\sinh \lambda z \,.\, J_m\lambda\rho \,.\, \sin m\phi$ vanishing at the whole of the external boundary of the space within the cylinder $\rho = 1$, the parallel planes $z = 0$, $z = \frac{\pi i}{\lambda}$, and the axial planes $\phi = 0$, $\phi = \frac{\pi}{m}$. But this is impossible, the case $\frac{\pi}{m} > 2\pi$ not excepted. Thirdly, $J_m\lambda$ has no pure imaginary zeros; otherwise we should have, as in the first part of the proof $J_m\lambda$, $J_{-m}\lambda$ vanishing simultaneously; that is J, G vanishing simultaneously, a supposition inconsistent with the fundamental relation

$$G_m\lambda \,.\, \frac{d}{d\lambda} J_m\lambda - J_m\lambda \frac{d}{d\lambda} G_m\lambda = \frac{1}{\lambda} .$$

(*b*) That the functions $G_m(\lambda)$ and $J_m a\lambda \,.\, G_m b\lambda - J_m b\lambda G_m a\lambda$ have no *complex* zeros is proved just as above; that they have no *real* zeros follows by considering the potentials

$$\sinh \lambda z \,.\, G_m\lambda\rho \,.\, \sin m\phi, \text{ with } \rho \gtrless 1 ;$$

and $\sinh \lambda z (J_m\lambda\rho G_m\lambda b - G_m\lambda\rho J_m\lambda b) \sin m\phi$, with ρ between a and b.

Each function has, however, an infinite number of purely imaginary zeros; this almost follows from the approximate formulas for m large which have already been given, or may be formally proved by the following considerations.

Take $G_m\lambda$ for example: this, as has been shown, is infinite for an infinite, not pure imaginary, m; but $m^2 G_m\lambda$ is infinite for every infinite m. Now it follows easily from the second formula of 4 (i) that any zero must be simple, as G, $\frac{dG}{dm}$ cannot vanish together. If, therefore, G have only a finite number of zeros, the uniform function $\frac{1}{m^2 G}$ would have only a finite number of accidental singularities; vanishing at infinity as it does, it would therefore be a rational function, a supposition excluded by the form of G at infinity.

Hence $G_m\lambda$ and similarly $J_m\lambda a G_m\lambda b - J_m\lambda b G_m\lambda a$ have an infinite number of pure imaginary zeros.

The relation we have made use of in (a) above, namely,

$$G_m\lambda\frac{d}{d\lambda}J_m\lambda - J_m\lambda\frac{d}{d\lambda}G_m\lambda = \frac{1}{\lambda},$$

is usually proved by the common method applicable to two solutions of a linear differential equation of the second order. It may also be proved geometrically, by a simple application of Green's Theorem, a method essentially the same as the former. But it is interesting to observe that it follows easily from the character of the left hand expression as a function of m; it is, namely, a holomorphic function whose form at infinity is $\frac{1}{\lambda}$; the function is therefore $\frac{1}{\lambda}$ for every m.

Spherical Harmonics.

6. If the potential function $(x+iy)^m$ be inverted from the point $(0, 0, -\xi)$ we obtain another potential function

$$(x+iy)^m/\{x^2+y^2+(z+\xi)^2\}^{m+\frac{1}{2}}.$$

Hence also
$$(x+iy)^m\int_0^\infty \frac{\xi^{n+n}d\xi}{\{\rho^2+(z+\xi)^2\}^{m+\frac{1}{2}}}$$

is a potential function, if the integral be convergent.

Change to polar coordinates r, θ, ϕ and put $\xi = r\lambda$, where λ is the new variable of integration.

The potential becomes

$$r^n\sin^m\theta e^{im\phi}\int_0^\infty \frac{\lambda^{m+n}d\lambda}{1+2\lambda\cos\theta+\lambda^2)^{m+\frac{1}{2}}},$$

and is therefore, by definition, a spherical harmonic of degree n, and rank m.

The function of θ represented by the integral may be expanded as follows.

We have $1+2\lambda\cos\theta+\lambda^2 = (1+\lambda)^2 - 2\lambda(1-\cos\theta)$

$$= (1+\lambda)^2\left\{1-\frac{4\lambda}{(1+\lambda)^2}\alpha\right\},$$

writing α for $\frac{1}{2}(1-\cos\theta)$ or $\frac{1}{2}(1-\mu)$

$$\therefore \int_0^\infty \frac{\lambda^{m+n}d\lambda}{(1+2\lambda\cos\theta+\lambda^2)^{m+\frac{1}{2}}} = \int_0^\infty \frac{\lambda^{m+n}}{(1+\lambda)^{2m+1}}\left\{1-\frac{4\lambda}{(1+\lambda)^2}\alpha\right\}^{-m-\frac{1}{2}}d\lambda.$$

If modulus $\alpha < 1$, the last factor can be expanded by the Binomial Theorem, and the series integrated term by term.

Coefficient of a^p will be

$$\frac{\Pi(m-\frac{1}{2}+p)}{\Pi(m-\frac{1}{2})\Pi p}\cdot 4^p\int_0^\infty \lambda^{m+n+p}(1+\lambda)^{-2m-2p-1}d\lambda$$

$$=\frac{\Pi(m-\frac{1}{2}+p)}{\Pi(m-\frac{1}{2})\Pi p}4^p\frac{\Pi(m+n+p)\Pi(m-n-1+p)}{\Pi(2m+2p)}.$$

Coefficient of a^{p+1} is obtained from this by multiplying by

$$\frac{m+p+\frac{1}{2}}{p+1}\cdot 4\cdot\frac{(m+n+p+1)(m-n+p)}{(2m+2p+1)(2m+2p+2)}$$

$$=\frac{(m+n+1+p)(m-n+p)}{(p+1)(m+p+1)}.$$

Hence the integral is

$$\frac{\Pi(m+n)\Pi(m-n-1)}{\Pi 2m}\left\{1+\frac{(m-n)(m+n+1)}{1\cdot(m+1)}a+\ldots\right\}$$

$$=\frac{\Pi(m+n)\Pi(m-n-1)}{\Pi 2m}\mathrm{F}\left(m-n,\ m+n+1,\ m+1,\ \frac{1-\mu}{2}\right)$$

$$=\frac{\Pi(m+n)\Pi(m-n-1)}{\Pi 2m}\left(\frac{1+\mu}{2}\right)^{-m}\mathrm{F}\left(n+1,\ -n,\ m+1,\ \frac{1-\mu}{2}\right),$$

where we use Euler's Theorem for the hypergeometric function,

$$\mathrm{F}(\alpha,\ \beta,\ \gamma,\ x)=(1-x)^{\gamma-\alpha-\beta}\mathrm{F}(\gamma-\alpha,\ \gamma-\beta,\ \gamma,\ x).$$

The function of θ

$$\frac{1}{\Pi m}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{m}{2}}\mathrm{F}\left(n+1,\ -n,\ m+1,\ \frac{1-\mu}{2}\right)$$

will be denoted by $\mathrm{P}_n^m(\mu)$.

When $m=0$, the above solid harmonic

$$\text{(i)}\quad \int_0^\infty \frac{\xi^n d\xi}{\{\rho^2+(z+\xi)^2\}^{\frac{1}{2}}}=\Pi n\,.\,\Pi(-n-1)r^n\mathrm{P}_n(\mu)$$

$$=\ -\ \frac{\pi}{\sin n\pi}\ r^n\mathrm{P}_n(\mu).$$

For the convergence of the definite integral we have used, the real parts of $m+n+1$, $m-n$, must be positive; this restriction still allows a continuous range of values for the variables m, n. But the definition of $\mathrm{P}_n^m(\mu)$ clearly defines a function of m, n as well as

of μ; the solid harmonic is also a function of m, n, as is likewise $\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}$ of the solid harmonic; this being zero for a continuous range of values of m, n, is zero for all values. P_n^m is clearly a holomorphic function both of m and of n.

As a function of μ, P_n^m will here be considered only for real values of μ from -1 to $+1$, this being all that is needed for the physical applications that follow. The definition exhibits sufficiently the behaviour of the function near $\mu=1$; a formula showing its character near $\mu=-1$ will be found presently.

We note here, as easily derived from the definite integral, the two formulæ

$$(m+n+1)P_{n+1}^m-\mu(2n+1)P_n^m+(n-m)P_{n+1}^m=0$$

and

$$P_n^m(0)=\frac{\sqrt{\pi}}{2^m}\cdot\frac{1}{\Pi\frac{m+n}{2}\cdot\Pi\frac{m-n-1}{2}}.$$

7. $P_n^m(\mu)$ is a solution of the differential equation

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dP}{d\mu}\right\}+\left\{n(n+1)-\frac{m^2}{1-\mu^2}\right\}P=0.$$

Since the equation is unaltered by changing μ into $-\mu$, or m into $-m$, we have three other solutions

$$P_n^{-m}(\mu),\quad P_n^m(-\mu),\quad P_n^{-m}(-\mu)$$

The two linear relations connecting the four solutions are found below. Change of n into $-n-1$ does not affect the equation, but this does not lead to any new solution, since the definition gives at once $P_n^m(\mu)=P_{-n-1}^m(\mu)$, an important relation.

Expression of the Harmonics by Bessel Functions.

8.

$\int_0^\infty e^{-\lambda z}J_m(\lambda\rho)\lambda^n d\lambda\,.\cos m\phi$ is a potential function, $(n+m>-1)$

Putting $z=r\cos\theta$, $\rho=r\sin\theta$, $\lambda r=\kappa$, we change the integral to

$$r^{-n-1}\cos m\phi\int_0^\infty e^{-\kappa\cos\theta}J_m(\kappa\sin\theta)\kappa^n d\kappa.$$

The function is, then, a solid spherical harmonic of degree $-n-1$, rank m.

Hence $$\int_0^\infty e^{-\kappa\cos\theta} J_m(\kappa\sin\theta)\kappa^n d\kappa = AP_n^m(\mu) + BP_n^{-m}(\mu).$$

To find the constants, suppose $m>0$; then $B=0$, since P_n^{-m} is infinite for $\theta=0$; further dividing by $\sin^m\theta$ and putting $\theta=0$,

$$\int_0^\infty \frac{e^{-\kappa}\kappa^{m+n}}{2^m \Pi m} d\kappa = A \cdot \frac{1}{2^m \Pi m}.$$

$$\therefore \quad A = \Pi(m+n)$$

and $$\int_0^\infty e^{-\lambda z} J_m \lambda\rho \,.\, \lambda^n d\lambda = \frac{\Pi(m+n)}{r^{n+1}} P_n^m(\cos\theta),$$

for all values of m, n, θ for which both sides of the equation retain a meaning.

Also $$\int_0^\infty e^{-\lambda\cos\theta} J_m(\lambda\sin\theta)\lambda^n d\lambda = \Pi(m+n) P_n^m(\cos\theta)$$
$$= p_n^m(\cos\theta), \text{ say.}$$

We have thus a representation of P by means of J, provided

$$\theta < \frac{\pi}{2}, \quad m+n > -1, \quad (i.e., \text{ real part of } m+n > -1).$$

The restriction upon m, n may at once be removed by taking a complex path of integration. Thus, supposing the λ plane cut from 0 to $-\infty$,

$$\int e^{\lambda z} J_m \lambda\rho \,.\, \lambda^n d\lambda = \frac{2\pi i}{\Pi(-m-n-1)} \frac{1}{r^{n+1}} P_n^m\left(\frac{z}{r}\right),$$

and $$\int e^{\lambda z} J_m \lambda\rho \,.\, \lambda^{-n-1} d\lambda = \frac{2\pi i}{\Pi(n-m)} \,.\, r^n P_n^m\left(\frac{z}{r}\right),$$

the path being from $-\infty$, under 0, round 0, back to $-\infty$, and the only restriction $z>0$.

We proceed to obtain a representation of P in which there is no restriction upon θ.

9. We may conveniently begin by explaining here a notation which will frequently be used in dealing with complex integrals. In the plane of any variable λ, let E, W be points on the positive and negative sides respectively of the real axis; N, S points on the positive and negative sides of the axis of imaginaries;

E, N, W, S thus corresponding to the points of the compass. These points will be supposed at an indefinitely great distance; and the symbol $\int f(\lambda)d\lambda \,\Big|\, \text{ON}$, for example, will be used to denote an integration over the whole of the upper half of the imaginary axis.

$\int f(\lambda)d\lambda \,\Big|\, \text{OE}$ is therefore the same as $\int_0^\infty f(\lambda)d\lambda$.

Consider now the integral

$$\int_0^\infty e^{i\lambda\cos\theta} i^m \mathrm{G}_m(i\lambda\sin\theta)\lambda^n d\lambda \,.\, (i^m \equiv e^{\frac{m\pi i}{2}}).$$

This may be written

$$\int e^{\lambda\cos\theta} i^m \mathrm{G}_m(\lambda\sin\theta) \,.\, \lambda^n d\lambda \,.\, i^{-n-1} \,\Big|\, \text{ON}$$

and the path may be deformed into OW, provided $\cos\theta > 0$, since $\mathrm{G}_m(\lambda\sin\theta)$ vanishes at infinity in the upper part of the plane.

Hence $\int_0^\infty e^{i\lambda\cos\theta} i^m \mathrm{G}_m(i\lambda\sin\theta)\lambda^n d\lambda$

$$= \int_0^\infty e^{-\lambda\cos\theta} i^m \mathrm{G}_m(-\lambda\sin\theta)\lambda^n d\lambda \,.\, i^{n+1}$$

$$= \frac{\pi}{2\sin m\pi} \,.\, i^{n+1} \int_0^\infty e^{-\lambda\cos\theta} (i^{-m}\mathrm{J}_{-m}\lambda\sin\theta - i^m \mathrm{J}_m \lambda\sin\theta)\lambda^n d\lambda$$

(i) $$= \frac{\pi}{2\sin m\pi} i^{n+1}\{i^{-m}p_n^{-m}(\mu) - i^m p_n^m(\mu)\}$$

Similarly $\int_0^\infty e^{-i\lambda\cos\theta} i^m \mathrm{G}_m(i\lambda\sin\theta)\lambda^n d\lambda$

$$= \int e^{-\lambda\cos\theta} i^m \mathrm{G}_m(\lambda\sin\theta)\lambda^n d\lambda \,.\, i^{-n-1} \,\Big|\, \text{ON}$$

$$= \int \,\Big|\, \text{OE, if } \cos\theta > 0$$

(ii) $$= \frac{\pi}{2\sin m\pi} i^{-n-1}\{i^m p_n^{-m}(\mu) - i^{-m} p_n^m(\mu)\}.$$

From these two relations

(iii) $$\int_0^\infty \sin\left\{\lambda\cos\theta + (m-n-1)\frac{\pi}{2}\right\} i^m \mathrm{G}_m(i\lambda\sin\theta)\lambda^n d\lambda$$

$$= -\frac{\pi}{2} p_n^m(\mu).$$

This formula extends the representation of the function to values of θ greater than $\frac{\pi}{2}$, provided the real parts of both $n+m$ and $n-m$ be greater than -1.

In (iii) we may change θ into $\pi-\theta$, or m into $-m$; neither of these changes, it may be observed, affects $i^m G_m(i\lambda\sin\theta)$.

In (i) change θ into $\pi-\theta$; the integral becomes the integral of (ii). Hence

(iv) $$i^{n+1-m}p_n^{-m}(-\mu)-i^{n+1+m}p_n^m(-\mu)=i^{m-n-1}p_n^{-m}(\mu)-i^{-m-n-1}p_n^m(\mu).$$

Similarly, by changing θ into $\pi-\theta$ in (ii),

(v) $$i^{m-n-1}p_n^{-m}(-\mu)-i^{-n-1-m}p_n^m(-\mu)=i^{-m+n+1}p_n^{-m}(\mu)-i^{m+n+1}p_n^m(\mu).$$

By means of these two equations, any two of the four functions

$$p_n^m(\mu),\quad p_n^m(-\mu),\quad p_n^{-m}(\mu),\quad p_n^{-m}(-\mu)$$

can be expressed in terms of the remaining two.

10. Eliminating $p_n^{-m}(\mu)$ from these equations, we find

$$\sin m\pi p_n^m(\mu)=\sin n\pi p_n^m(-\mu)+\sin(m-n)\pi p_n^{-m}(-\mu),$$

or returning to the P functions,

(i) $$\Pi(m+n)\sin m\pi \mathrm{P}_n^m(\mu)$$
$$=\Pi(m+n)\sin n\pi \mathrm{P}_n^m(-\mu)+\Pi(n-m)\sin(m-n)\pi \mathrm{P}_n^{-m}(-\mu).$$

This equation, and that obtained from it by changing μ into $-\mu$, may be regarded as the linear relations connecting the four solutions referred to in §7.

Since $$\Pi(n-m)\sin(m-n)\pi=\pi/\Pi(m-n-1),$$

we may write (i) in the form

(ii) $$\pi \mathrm{P}_n^{-m}(\mu)=\Pi(m+n)\Pi(m-n-1)\{\sin m\pi \mathrm{P}_n^m(-\mu)-\sin n\pi \mathrm{P}_n^m(\mu)\}.$$

The equation (i) or (ii) has many important applications, of which one or two will be pointed out.

(a) The equation gives the form of $\mathrm{P}_n^m(\mu)$ in the neighbourhood of $\mu=-1$, or of $\mathrm{P}_n^m(-\mu)$ near $\mu=+1$.

When m is an integer, the method of limits has to be applied, as in the case of the Bessel function $G_m(x)$ and

the expansion will involve a logarithm, unless n also be an integer.

(b) When $n-m$ is a positive integer p, we have

$$\Pi(m+n)\sin m\pi \mathrm{P}_n^m(\mu) = \Pi(m+n)(-)^p \sin m\pi \mathrm{P}_n^m(-\mu)$$

so that if m be positive and not integral,

$$\mathrm{P}_{m+p}^m(\mu) = (-)^p \mathrm{P}_{m+p}^m(-\mu).$$

But the two sides of this equation are, for a given p, continuous functions of m, so that this holds for every m.

The function $\mathrm{P}_{m+p}^m(\mu)$ is important; it is the product of $(1-\mu^2)^{\frac{m}{2}}$ by a rational integral function of μ, and for m positive vanishes both at $\mu=1$ and $\mu=-1$.

We have, in fact, by definition,

$$\mathrm{P}_n^m(\mu) = \frac{1}{2^m \Pi m}(1-\mu^2)^{\frac{m}{2}} \mathrm{F}\left(m-n,\ m+n+1,\ m+1,\ \frac{1-\mu}{2}\right)$$

and the series terminates when $m-n$ is zero or a negative integer.

(c) When m is an integer, we similarly obtain

$$\Pi(n+m)\mathrm{P}_n^m(\mu) = (-)^m \Pi(n-m)\mathrm{P}_n^{-m}(\mu).$$

If now n also is taken integral, then if $n-m<0$, $\mathrm{P}_n^{-m}(\mu)=0$.

Some Fundamental Integrals.

11. Let u, v be two harmonics of degrees, n, p and ranks m, l, so that

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{du}{d\mu}\right\} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}u = 0$$

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dv}{d\mu}\right\} + \left\{p(p+1) - \frac{l^2}{1-\mu^2}\right\}v = 0.$$

Multiply these by v, u; subtract, and integrate from h to k.

$$\therefore \int_h^k \left\{(n-p)(n+p+1) + \frac{l^2-m^2}{1-\mu^2}\right\} uv\,d\mu$$
$$= \left[(1-\mu^2)\left(u\frac{dv}{d\mu} - v\frac{du}{d\mu}\right)\right]_h^k$$

This equation has applications of precisely the same sort as the

corresponding equation for the Bessel. Into the details of these it is not proposed to enter, but one or two results will be given.

(i) Take $u = \mathrm{P}_n^m(\mu)$, $v = \mathrm{P}_n^{-m}(\mu)$.

Then $\left| (1-\mu^2)\left(u\frac{dv}{d\mu} - v\frac{du}{d\mu}\right) \right|_h^k = 0$

i.e., $(1-\mu^2)\left(u\frac{dv}{d\mu} - v\frac{du}{d\mu}\right)$ is independent of μ; to find its value, we may take the limit for $\mu = 1$.

Now near $\mu = 1$, we have $u = \frac{1}{2^m \Pi m}(1-\mu^2)^{\frac{m}{2}}$

$$v = \frac{1}{2^{-m}\Pi - m}(1-\mu^2)^{-\frac{m}{2}}.$$

Hence we find $(1-\mu^2)\left\{\mathrm{P}_n^m(\mu)\frac{d}{d\mu}\mathrm{P}_n^{-m}(\mu) - \mathrm{P}_n^{-m}(\mu)\frac{d}{d\mu}\mathrm{P}_n^m(\mu)\right\}$

$$= \frac{2}{\pi}\sin m\pi.$$

From this again, by means of 10 (ii)

we readily deduce $(1-\mu^2)\left\{\mathrm{P}_n^m(\mu)\frac{d}{d\mu}\mathrm{P}_n^m(-\mu) - \mathrm{P}_n^m(-\mu)\frac{d}{d\mu}\mathrm{P}_n^m(\mu)\right\}$

$$= \frac{2}{\Pi(m+n)\Pi(m-n-1)}.$$

(ii) Take $u = \mathrm{P}_n^m(\mu)$, $v = \mathrm{P}_p^m(-\mu)$; also suppose real part of m positive.

Then $\int_{-1}^{+1} \mathrm{P}_n^m(\mu)\mathrm{P}_p^m(-\mu)d\mu$

$$= \frac{1}{(n-p)(n+p+1)}\left[(1-\mu^2)\left(u\frac{dv}{d\mu} - v\frac{du}{d\mu}\right)\right]_{-1}^{+1}$$

From 10 (ii), near $\mu = 1$, we have

$$\mathrm{P}_p^m(-\mu) = \frac{\pi}{\sin m\pi}\frac{1}{\Pi(m+p)\Pi(m-p-1)}\cdot\frac{2^m}{\Pi(-m)}(1-\mu^2)^{-\frac{m}{2}}$$

and therefore

$$\left| (1-\mu^2)\left(u\frac{dv}{d\mu} - v\frac{du}{d\mu}\right) \right|_{\mu=1} = \frac{2}{\Pi(m+p)\Pi(m-p-1)}$$

Similarly

$$\left| (1-\mu^2)\left(u\frac{dv}{d\mu} - v\frac{du}{d\mu}\right) \right|_{\mu=-1} = \frac{2}{\Pi(m+n)\Pi(m-n-1)}.$$

Hence $\int_{-1}^{+1} P_n^m(\mu) P_p^m(-\mu) d\mu$

$$= \frac{2}{(n-p)(n+p+1)}\left[\frac{1}{\Pi(m+p)\Pi(m-p-1)} - \frac{1}{\Pi(m+n)\Pi(m-n-1)}\right]$$

a result interesting as the generalisation of various known theorems.

In special cases, the value of the dexter has to be found by limits.

By far the most important case is that where $n-m$, $p-m$ are zero or positive integers. Since $P_n = P_{-n-1}$ we may suppose $n+\frac{1}{2}$ and $p+\frac{1}{2}$ not less than 0.

Then the equation shows that if n, p are different, the integral vanishes, since $\Pi(m-p-1)$, $\Pi(m-n-1)$ are infinite.

In order to find the value of the integral when $n=p$, put $p=m+q$, q zero or a positive integer;
and find the limit as the single variable n approaches the fixed $m+q$.

We have, the integral

$$= \frac{1}{\pi} \frac{\sin(n-m)\pi}{n-q-m} \cdot \frac{2}{n+q+m+1} \cdot \frac{\Pi(n-m)}{\Pi(n+m)} \cdot$$

$$= (-)^q \frac{2}{2m+2q+1} \cdot \frac{\Pi q}{\Pi(m+2q)}, \quad \text{in the limit.}$$

But, q being an integer,

$$P_{m+q}^m(-\mu) = (-)^q P_{m+q}^m(\mu), \quad [10\ (b)].$$

Hence we may state the result: when $n-m$ is a positive integer or zero,

$$\int_{-1}^{+1} \{P_n^m(\mu)\}^2 d\mu = \frac{2}{2n+1} \cdot \frac{\Pi(n-m)}{\Pi(n+m)} \cdot$$

12. The harmonic $P_n^m(\mu)$ as a function of m, its rank.

The function is holomorphic; its form for m infinite is given by the definition $P_n^m(\mu) = \frac{1}{\Pi m}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{m}{2}} F\left(n+1, -n, m+1, \frac{1-\mu}{2}\right)$.

For m very large, this gives $P^m = \frac{1}{\Pi m}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{m}{2}}$.

As to the zeros of the function, the important case is that in which the given n is of the form $-\frac{1}{2}+\lambda i$, λ real.

By methods altogether similar to those used for the Bessels, it may be proved, $(n = -\frac{1}{2} + \lambda i)$.

(i) $\mathrm{P}_n^m(\mu)$ has no zeros for which the real part of m is positive or zero.

(ii) The function

$$\Pi(m+n)\Pi(m-n-1)\{\mathrm{P}_n^m(h)\mathrm{P}_n^m(-k) - \mathrm{P}_n^m(k)\mathrm{P}_n^m(-h)\},$$

which, by 10 (ii), is the same as

$$\frac{\pi}{\sin m\pi}\{\mathrm{P}_n^m(h)\mathrm{P}_n^{-m}(k) - \mathrm{P}_n^m(k)\mathrm{P}_n^{-m}(h)\},$$

and is therefore even and holomorphic, has an infinite number of purely imaginary simple zeros.

When m is very large, this function has the form

$$\frac{2}{m}\sinh\{m\log(\tan\alpha\cot\beta)\}\,; \quad h = \cos\alpha\,, \quad k = \cos\beta.$$

13. THE HARMONIC $\mathrm{P}_n^m(\mu)$ AS A FUNCTION OF n, ITS DEGREE.

As to the form of the function for n infinite, we have room here for only a few statements; for various investigations bearing on the question, reference may be made to the memoir of Dr Hobson, *l.c.*, and to Heine, I. p. 178, II. p. 223.

In the first place, if $\frac{\pi}{6} < \theta < \frac{5\pi}{6}$, we have

(i) $$\mathrm{P}_n^m(\cos\theta) = \frac{1}{\sqrt{2\pi\sin\theta}}\frac{\Pi(n-m)}{\Pi(n+\frac{1}{2})}$$
$$\left[\begin{array}{l} i^{m+\frac{1}{2}}e^{-(n+\frac{1}{2})i\theta}\mathrm{F}\left(\frac{1}{2}-m,\ \frac{1}{2}+m,\ n+\frac{3}{2},\ \frac{ie^{-i\theta}}{2\sin\theta}\right) \\ +i^{-(m+\frac{1}{2})}e^{+(n+\frac{1}{2})i\theta}\mathrm{F}\left(\frac{1}{2}-m,\ \frac{1}{2}+m,\ n+\frac{3}{2},\ \frac{-ie^{i\theta}}{2\sin\theta}\right) \end{array}\right]$$

but the series diverge for values of θ outside the limits specified. For any value of θ, not 0 or π, however, it is possible to show that the limiting form of the function is given by the first terms of these series, so that for any n with large modulus, and phase not equal to π

(ii) $$\mathrm{P}_n^m(\cos\theta) = \frac{1}{\sqrt{2\pi\sin\theta}}\frac{\Pi(n-m)}{\Pi(n+\frac{1}{2})}\left\{\begin{array}{l} i^{m+\frac{1}{2}}e^{-(n+\frac{1}{2})i\theta} \\ +i^{-(m+\frac{1}{2})}e^{+(n+\frac{1}{2})i\theta} \end{array}\right\}.$$

It is very curious that if in the equation,

$$-\frac{\pi}{2}\Pi(m+n)P_n^m(\cos\theta) = \int_0^\infty \sin\left\{\lambda\cos\theta+(m-n-1)\frac{\pi}{2}\right\} i^m G_m(i\lambda\sin\theta)\lambda^n d\lambda$$

we substitute for G its semi-convergent expansion and integrate term by term (a process not, of course, defensible) we obtain

$$\text{(iii)}\quad P_n^m(\cos\theta) = \frac{1}{\sqrt{2\pi\sin\theta}}\frac{\Pi(n-\frac{1}{2})}{\Pi(n+m)}$$

$$\left[\begin{array}{l} i^{m+\frac{1}{2}}e^{-(n+\frac{1}{2})i\theta}F\left(\frac{1}{2}-m,\ \frac{1}{2}+m,\ \frac{1}{2}-n,\ -\frac{ie^{i\theta}}{2\sin\theta}\right) \\ +i^{-(m+\frac{1}{2})}e^{+(n+\frac{1}{2})i\theta}F\left(\frac{1}{2}-m,\ \frac{1}{2}+m,\ \frac{1}{2}-n,\ \frac{ie^{-i\theta}}{2\sin\theta}\right) \end{array}\right].$$

If n be replaced by $-n-1$, the hypergeometic series are those of (i), and the interesting question suggests itself, does the right hand of (iii) actually represent the function P for the range of values of θ for which it has a meaning?

For a large n, (i) and (iii) give the same limiting form for P, but on examination it appears that the dexter members are not really identical, unless m is half an odd integer, in which case, it may be noticed, the series terminate, like the series for J_m in the like case.

14. With regard to the zeros of P as a function of n, it is only necessary to consider the case of m real and positive, or zero. Then

(i) $P_n^m(\cos\alpha)$ has an infinite number of real, simple zeros, and no others.

For a large n, we have from (ii) of last paragraph

$$P_n^m(\cos\theta) = \frac{\sqrt{2}}{\sqrt{\pi\sin\theta}}\frac{\Pi(n-m)}{\Pi(n+\frac{1}{2})}\left\{\cos\left[(n+\tfrac{1}{2})\theta-(m+\tfrac{1}{2})\frac{\pi}{2}\right]\right\}.$$

(ii) The function

$$\Pi(m+n)\Pi(m-n-1)\{P_n^m(\cos\alpha)P_n^m(-\cos\beta)-P_n^m(\cos\beta)P_n^m(-\cos\alpha)\},$$

or the same thing,

$$\frac{\pi}{\sin m\pi}\{P_n^m(\cos\alpha)P_n^{-m}(\cos\beta)-P_n^m(\cos\beta)P_n^{-m}(\cos\alpha)\}$$

has also an infinite number of real, simple zeros, and no others.

For a large n, this function

$$= \frac{2}{n\sqrt{\sin\alpha \sin\beta}} \sin\{(n+\tfrac{1}{2})(\alpha-\beta)\}.$$

15. We have now considered the limiting forms of $P_n^m(\mu)$ in three cases when

(i) μ approaches -1, with n, m fixed
(ii) m approaches ∞, with n, μ fixed
(iii) n approaches ∞, with m, μ fixed.

Two important cases remain, in which one only of the variables is held fast, while the other two vary, subject to a constant relation.

First, let n tend to infinity, θ to zero, while the product $n\theta$ remains constant $= \lambda$, suppose.

Then from the definition of P, we obtain, in a well-known way,

$$\text{Limit } n^m P_n^m(\cos\theta) = J_m(\lambda).$$

Supposing now, that not merely modulus n, but the *imaginary* part of n tends to infinity (remaining positive), we have from 10 (ii)

$$\Pi(m+n)\Pi(m-n-1)\sin m\pi P_n^m(-\mu)$$
$$= \pi P_n^{-m}(\mu) + \sin n\pi\, \Pi(m+n)\Pi(m-n-1)P_n^m(\mu)$$
$$= \pi P_n^{-m}(\mu) - \frac{\pi \sin n\pi}{\sin(n-m)\pi} \quad \frac{\Pi(n+m)}{\Pi(n-m)} \quad P_n^m(\mu)$$

or since $\Pi(n+m) = n^m \Pi n$ approximately

$$n^{-m}\Pi(m+n)\Pi(m-n-1)P_n^m(-\mu)$$
$$= \frac{\pi}{\sin m\pi}\{n^{-m}P_n^{-m}(\mu) - e^{-im\pi}n^m P_n^m(\mu)\}$$

or Limit $n^{-m}\Pi(m+n)\Pi(m-n-1)P_n^m(-\mu)$

$$= \frac{\pi}{\sin m\pi}\{J_{-m}(\lambda) - e^{-im\pi}J_m(\lambda)\}$$
$$= 2G_m(\lambda),$$

where the imaginary part of λ is positive.

Second, while θ remains fixed, let m and n tend to infinity, so that $n - m = p$, a given positive integer.

Since $P_n^m(\mu) = \frac{1}{2^m \Pi m}(1-\mu^2)^{\frac{m}{2}} F\left(m-n,\ m+n+1,\ m+1,\ \frac{1-\mu}{2}\right)$

we have, in this case

$$P_n^m(\mu) = \frac{1}{2^m \Pi m}(1-\mu^2)^{\frac{m}{2}}\left\{1 - \frac{p}{1}(1-\mu) + \frac{p(p-1)}{1.2}(1-\mu)^2, \text{ etc.}\right\}$$

approximately.

So that $\underset{m=\infty}{\text{Limit}} \quad P_{m+p}^m(\mu) = \frac{1}{2^m \Pi m}(1-\mu^2)^{\frac{m}{2}}\mu^p.$

Addition Theorem.

16. The distance D between two points (r, α, ϕ), (r', β, ϕ') is given by $D^2 = r^2 - 2rr'\cos\gamma + r'^2$, where

$$\cos\gamma = \cos\alpha\cos\beta + \sin\alpha\sin\beta\cos(\phi - \phi').$$

$P_n(-\cos\gamma)$ is, if $\gamma > 0$, a continuous periodic function of $\phi - \phi'$, which by Fourier's Theorem can be expanded in the form

$$P_n(-\cos\gamma) = A_0 + A_1\cos(\phi - \phi') + \ldots + A_m\cos m(\phi - \phi') + \ldots$$

with $$A_m\cos m\phi' = \frac{1}{\pi}\int_0^{2\pi} P_n(-\cos\gamma)\cos m\phi d\phi.$$

The definite integral may be determined as follows.

Consider the potential V, given by

$$V = r^n\cos m\phi\, P_n^m(\cos\theta)\, P_n^m(-\cos\alpha); \quad 0 < \theta < \alpha$$
$$V = r^n\cos m\phi\, P_n^m(\cos\alpha)\, P_n^m(-\cos\theta); \quad \alpha < \theta < \pi$$

where m is a positive integer, and $n = -\frac{1}{2} + \lambda i$, λ real.

This potential can be produced by a distribution of matter on the cone $\theta = \alpha$, whose density at r, α, ϕ is σ,

given by $$4\pi\sigma = \frac{1}{r}\left|\frac{dV}{d\theta}\right|_{\theta=\alpha-0}^{\theta=\alpha+0}$$

$$= r^{n-1}\cos m\phi \left\{\begin{matrix} P_n^m(-\cos\alpha)\frac{d}{d\theta}P_n^m(\cos\theta) \\ -P_n^m(\cos\alpha)\frac{d}{d\theta}P_n^m(-\cos\theta)\end{matrix}\right\}\theta = \alpha.$$

$$= r^{n-1}\cos m\phi \cdot \frac{2}{\sin\alpha} \cdot \frac{1}{\Pi(m+n)\Pi(m-n-1)}$$

from 11 (i).

The potential of this distribution at (r', β, ϕ') is

$$\iint \frac{\sigma dS}{D}$$

$$= \frac{1}{4\pi}\int_0^{2\pi} \cos m\phi d\phi \int_0^{\infty} \frac{2}{\Pi(m+n)\Pi(m-n-1)} \frac{r^n dr}{\sqrt{(r^2 - 2rr'\cos\gamma + r'^2)}}$$

$$= \frac{1}{2\pi} \frac{\Pi(n)\Pi(-n-1)(r')^n}{\Pi(m+n)\Pi(m-n-1)} \int_0^{2\pi} \cos m\phi \,.\, P_n(-\cos\gamma) d\phi,$$

from 6 (i).

But the potential is, if $\alpha > \beta$, $r'^n \cos m\phi' P_n^m(\cos\beta) P_n^m(-\cos\alpha)$

$$\therefore \quad A_m = \frac{1}{\pi}\int_0^{2\pi} \cos m\phi \, P_n(-\cos\gamma) d\phi / \cos m\phi'$$

$$= 2\frac{\Pi(m+n)\Pi(m-n-1)}{\Pi n \,.\, \Pi(-n-1)} P_n^m(\cos\beta) P_n^m(-\cos\alpha),$$

or half of this, if $m = 0$.

Hence

(1) $P_n(-\cos\gamma)$

$$= P_n(\cos\beta)P_n(-\cos\alpha) + 2\sum_{m=1} c_m P_n^m(\cos\beta) P_n^m(-\cos\alpha)\cos m(\phi - \phi').$$

where $\alpha > \beta$; $c_m = \dfrac{\Pi(m+n)\Pi(m-n-1)}{\Pi n \,.\, \Pi(-n-1)}$

$$= (-)^m (n+m)(n+m-1)\ldots(n-m+1).$$

For m very large,

$$c_m P_n^m(\cos\beta) P_n^m(-\cos\alpha) = \frac{1}{m}\left(-\frac{\sin n\pi}{\pi}\right) \tan^m\frac{\beta}{2} \cot^m\frac{\alpha}{2}.$$

The series therefore converges absolutely if $\alpha > \beta$; it still converges if $\alpha = \beta$, provided $\phi - \phi'$ is not zero.

The series represents a function of n, which for a continuous range of values of n, has been proved equal to the function $P_n(-\cos\gamma)$; the series therefore represents this function for all values of n.

(2) Now change α into $\pi - \alpha$, $\phi - \phi'$ into $\pi - (\phi - \phi')$; we obtain

$$P_n\{\cos\alpha\cos\beta + \sin\alpha\sin\beta\cos(\phi - \phi')\}$$

$$= 2\sum_m{}'(-)^m c_m P_n^m(\cos\alpha) P_n^m(\cos\beta)\cos m(\phi - \phi'),$$

the restriction becoming $\pi > \alpha + \beta$.

(3) Again change β into $\pi - \beta$, $\phi - \phi'$ into $\pi - (\phi - \phi')$;

hence $$P_n\{\cos\alpha\cos\beta + \sin\alpha\sin\beta\cos(\phi - \phi')\}$$
$$= 2\sum_m{}'(-)^m c_m P_n^m(-\cos\alpha)P_n^m(-\cos\beta)\cos m(\phi - \phi');$$

with the restriction $\pi < \alpha + \beta$.

Of these three forms, the one originally obtained is the most important.

It is to be observed that when n is a positive integer,

$$\text{and} \quad m > n, \quad c_m = 0.$$

Many formulæ are shortened by the introduction of an additional function S, defined by the equation

$$S_n^m(\mu) \equiv \Pi(m+n)\Pi(m-n-1)P_n^m(-\mu)$$

The equation (1), for instance, may be written

(4) $$S_n(\cos\gamma) = 2\sum_m{}' S_n^m(\cos\alpha) \cdot P_n^m(\cos\beta)\cos m(\phi - \phi').$$
$$\alpha > \beta.$$

17. Addition Theorems involving harmonics of pure imaginary rank.

The well-known addition theorem for the Bessel function G is

(1) $$G_0(\lambda R) = G_0\lambda a \cdot J_0\lambda b + 2\Sigma G_m\lambda a \cdot J_m\lambda b \cdot \cos m(\phi - \phi')$$
$$= 2\sum_m{}' G_m\lambda a \cdot J_m\lambda b \cdot \cos m(\phi - \phi');$$

where $R = \sqrt{\{a^2 + b^2 - 2ab\cos(\phi - \phi')\}}$; and a, b are real and positive with $a > b$.

The series converges absolutely, and the theorem holds, whatever be the phase of λ. Hence taking the increment of the two members of the equation when the phase of λ increases by 2π, we deduce the addition theorem for J,

(2) $$J_0(\lambda R) = 2\sum_m{}' J_m\lambda a J_m\lambda b \cos m(\phi - \phi').$$

The series for $G_0(i\lambda R)$ may be transformed into an integral as follows.

Consider the function of m,

$$\frac{\cos m(\pi - \phi + \phi')}{\sin m\pi} G_m i\lambda a \cdot J_m i\lambda b,$$

in which we suppose a, b, λ real and positive, and

$$0 < \phi - \phi' < 2\pi.$$

Then $\cos m(\pi - \phi + \phi')/\sin m\pi$ vanishes for m infinite, unless m is real. Also (§ 3) for a large m,

$$i^{-m}\mathrm{J}_m i\lambda b = \frac{(\lambda b)^m}{2^m \Pi m}$$

$$i^m \mathrm{G}_m i\lambda a = \frac{\Pi m}{2m}\left(\frac{2}{\lambda a}\right)^m, \text{ if real part of } m \text{ be positive,}$$

and therefore $\mathrm{G}_m i\lambda a \,.\, \mathrm{J}_m i\lambda b = \frac{1}{2m}\left(\frac{b}{a}\right)^m$

For a pure imaginary m, modulus $\mathrm{G}_m i\lambda a \,.\, \mathrm{J}_m i\lambda b = 1/\text{modulus } m$. Hence $\mathrm{G}_m i\lambda a \,.\, \mathrm{J}_m i\lambda b \,.\, \cos m(\pi - \phi + \phi')/\sin m\pi$ vanishes for m infinite with real part positive or zero. (The vanishing is *effective*, by which we mean that, even when multiplied by m, the function tends to zero; a condition sufficient to ensure the evanescence of the integral of the function taken over any part of the circle at an infinite distance. But in cases like this, we often, for the sake of brevity, omit to call attention to this point explicitly, as it cannot fail to be observed.)

Take now a path of integration from S to N in the m plane. This path must not pass through O, which is a pole of the function; we therefore describe a small circle about O as centre, and take a path from S to N along the axis of imaginaries, but passing O by the semicircle, first, on the eastern side, second, on the western side.

The first integral

$= (-2\pi i)$ (sum of residues at poles to the right of O).

The second integral

$= (-2\pi i)$ (sum of the same residues + residue at O).

Since $i^m \mathrm{G}_m i\lambda a$ is an even function of m, we may write the sum of the integrals over the straight portions in the form

$$2\int \frac{\cos m(\pi - \phi + \phi')}{\sin m\pi}(i^m \mathrm{G}_m i\lambda a)(i^{-m}\mathrm{J}_m i\lambda b - i^m \mathrm{J}_{-m} i\lambda b)dm,$$

path from the north point of the small circle to N. We now diminish the small circle indefinitely, so that its contributions to the integrals neutralize each other, and obtain

$$2\int \frac{\cos m(\pi - \phi + \phi')}{\sin m\pi}\mathrm{G}_m i\lambda a(-e^{im\pi})\frac{2\sin m\pi}{\pi}\mathrm{G}_m i\lambda b\, dm \;\Big|\; \text{ON}$$

$$= -2\pi i \,.\, \frac{2}{\pi} \,.\, \Sigma' \cos m(\phi - \phi')\mathrm{G}_m i\lambda a\, \mathrm{J}_m i\lambda b$$

or

$$(3)\quad G_0(i\lambda R) = \frac{2}{\pi}\int_0^\infty \cosh s(\pi - \phi + \phi')e^{-s\pi} G_{is}i\lambda a \,.\, G_{is}i\lambda b \,.\, ds.$$

$$o < \phi - \phi' < 2\pi\,;\ \lambda,\ a,\ b \text{ real and positive.}$$

Similarly may be treated the expansion of $S_n(\cos\gamma)$, 16 (4).

For, when m is large

$$S_n^m \cos\alpha \,.\, P_n^m \cos\beta = \frac{1}{m}\left(\tan\frac{\beta}{2} \Big/ \tan\frac{\alpha}{2}\right)^m,$$

which vanishes when the real part of m is positive, and $\alpha > \beta$.

We find, just as above,

$$(4)\quad S_n(\cos\gamma)$$

$$= i\int_0^\infty \frac{\cosh s(\pi - \phi + \phi')}{\sinh s\pi}(S_n^{is}\cos\alpha \,.\, P_n^{is}\cos\beta - S_n^{-is}\cos\alpha \,.\, P_n^{-is}\cos\beta)ds$$

$$o < \phi - \phi' < 2\pi.$$

The factor of the integrand involving the S and P functions does not appear to be expressible in the form of the product of a function of α by the same function of β; but it is important to observe that this factor is really symmetrical in α and β.

For (§ 12) $S_n^m k \,.\, P_n^m h - S_n^m h \,.\, P_n^m k$ is even in m,

that is, equal to $S_n^{-m} k\, P_n^{-m} h - S_n^{-m} h\, P_n^{-m} k$.

Hence, transposing, we have

$$S_n^m k\, P_n^m h - S_n^{-m} k\, P_n^{-m} h = S_n^m h\, P_n^m k - S_n^{-m} h\, P_n^{-m} k.$$

Green's Function.

We proceed now to obtain Green's function in terms of harmonic potentials, for spaces bounded by surfaces of the cylindrical and spherical polar systems.

The bounding surfaces are complete, or, at any rate, if they be supposed completed, they do not intersect the space considered. It has not been thought necessary to give the solution for every variety of space satisfying this condition, but the cases omitted can easily be treated after the same manner as the others.

The coordinates of the pole are expressed by accented, and of the variable point by unaccented letters; these two points may, it is well known, be interchanged without altering the value of the function.

The reciprocal of the distance between the points is denoted by T, and the Green's function by V.

Cylindrical Coordinates.

$$z,\ \rho,\ \phi\,;\quad \mathrm{R}=\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}.$$

18. *Whole of space.* The Green's function is simply T.

(1) From 8

$$\mathrm{T}=\int_0^\infty e^{-\lambda(z-z')}\mathrm{J}_0\lambda\mathrm{R}\,.\,d\lambda$$

$$=2\int_0^\infty e^{-\lambda(z-z')}d\lambda\,\Sigma'_m\,\mathrm{J}_m\lambda\rho\,.\,\mathrm{J}_m\lambda\rho'.\cos m(\phi-\phi'). \qquad z>z'$$

(2) From 9 (iii)

$$\mathrm{T}=\frac{2}{\pi}\int_0^\infty \cos\lambda(z-z')\mathrm{G}_0 i\lambda\mathrm{R}\,.\,d\lambda$$

$$=\frac{4}{\pi}\int_0^\infty \cos\lambda(z-z')d\lambda\,\Sigma'_m\,\mathrm{G}_m i\lambda\rho\,.\,\mathrm{J}_m i\lambda\rho'.\cos m(\phi-\phi'). \qquad \rho>\rho'.$$

(3) The above forms for T are familiar; the first, which we may call the z form, is formally discontinuous at the z surface through the pole, the second, or ρ form, at the ρ surface through the pole. The third, or ϕ form, does not seem to have been given before. It is, from 18 (3)

$$\mathrm{T}=\frac{2}{\pi}\int_0^\infty \cos\lambda(z-z')\mathrm{G}_0 i\lambda\mathrm{R}\,.\,d\lambda$$

$$=\frac{4}{\pi^2}\int_0^\infty \cos\lambda(z-z')d\lambda\int_0^\infty \cosh s(\pi-\phi+\phi')e^{-s\pi}\mathrm{G}_{is}i\lambda\rho\,.\,\mathrm{G}_{is}i\lambda\rho'\,ds. \qquad 0<\phi-\phi'<2\pi.$$

19. *Space bounded by two parallel planes*

$$z=0,\quad z=c>0.$$

(1) The Green's function is obtained by adding to T a potential non-singular throughout the space, and equal to $(-\mathrm{T})$ on the boundary.

Using the z form for T, we see that this complementary potential must take the values

$$-\int_0^\infty e^{-\lambda(c-z')}J_0\lambda R d\lambda \text{ on } z=c, \text{ and}$$

$$-\int_0^\infty e^{-\lambda z'} \quad J_0\lambda R d\lambda \text{ on } z=0.$$

Such a potential is

$$-\int_0^\infty \left\{\frac{\sinh \lambda z}{\sinh \lambda c}e^{-\lambda(c-z')} + \frac{\sinh \lambda(c-z)}{\sinh \lambda c}e^{-\lambda z'}\right\}J_0\lambda R d\lambda,$$

for the integral converges absolutely so long as

$z+z'-2c<0$, $z+z'>0$, conditions necessarily fulfilled.

To obtain V, add T, which when $z>z'$, is

$$\int_0^\infty e^{-\lambda(z-z')}J_0\lambda R d\lambda.$$

Thus $$V=2\int_0^\infty \frac{\sinh \lambda(c-z)\sinh \lambda z'}{\sinh \lambda c}J_0\lambda R \,.\, d\lambda\,;\ z>z'.$$

For $z<z'$, interchange z and z'.

There may seem to be a certain loss of simplicity in uniting the part of the Green's function which contains the singularity, namely, T, to the other part of the function (often, by itself, with sign changed, called the Green's function) which is non-singular. This, however, is an essential step in the transformations we propose to make. Moreover, the form given for V has the advantage of showing at a glance that V vanishes for $z=c$, and (since for $z<z'$, we interchange z and z') for $z=0$.

(2) Since $$J_0(\lambda R)=\frac{1}{\pi i}\{G_0(\lambda R)-G_0(-\lambda R)\},\ R>0,$$

we may write

$$V=\frac{2}{\pi i}\int \frac{\sinh \lambda(c-z)\,.\,\sinh \lambda z'}{\sinh \lambda c}G_0\lambda R \,.\, d\lambda \;\Big|\; \text{WOE},$$

the function multiplying $G_0\lambda R$ being *odd* in λ.

The integrand effectively vanishes in the upper part of the λ plane, and therefore the integral equals $2\pi i$ into sum of residues at the poles on ON.

Hence $$V=\frac{4}{c}\sum_p \sin p\pi z/c \,.\, \sin p\pi z'/c \,.\, G_0 ip\pi R/c.$$

$$R>0.$$

This series converges with striking rapidity, except very near the pole, the rapidity increasing with R.

Since $G_0 ix = \sqrt{\pi/2x}\, e^{-x}$ for a large real x we have, for the limiting form

$$V = \frac{4}{c} \sin\frac{\pi z}{c} \cdot \sin\frac{\pi z'}{c} \cdot \sqrt{\frac{c}{2R}}\, e^{-\pi R/c},$$

a sufficient approximation if R is not less than, say, four or five times c.

Using the ρ form for $G_0 i\lambda R$, 17 (2),

$$V = 8/c \sum_p \sin\lambda z \sin\lambda z' \sum_m{}' G_m i\lambda\rho \cdot J_m i\lambda\rho' \cdot \cos m(\phi - \phi').$$

$$\lambda = p\pi/c, \quad \rho > \rho'.$$

(3) Using the ϕ form for $G_0 i\lambda R$, 17 (3),

$$V = 8/\pi c \cdot \sum_p \sin\lambda z \sin\lambda z' \int_0^\infty e^{-s\pi} \cosh s(\pi - \phi + \phi') G_{is} i\lambda\rho \cdot G_{is} i\lambda\rho' ds.$$

20. *Space bounded externally by a cylinder*

$$\rho = a.$$

(1) The ρ form for T is

$$T = \frac{4}{\pi} \int_0^\infty \cos\lambda(z - z') d\lambda \sum_m{}' G_m i\lambda\rho \cdot J_m i\lambda\rho' \cos m(\phi - \phi').$$

$$\rho > \rho'$$

Hence

$$V = \frac{4}{\pi} \int_0^\infty \cos\lambda(z - z') d\lambda \cdot \sum_m{}' \frac{J_m i\lambda\rho'}{J_m i\lambda a} \cdot$$

$$(J_m i\lambda a\, G_m i\lambda\rho - J_m i\lambda\rho\, G_m i\lambda a) \cos m(\phi - \phi').$$

$$\rho > \rho'.$$

(2) In the ρ form for T we may interchange the order summation, integration; or, in other words, integrate term by term. The proof is in this case easy, for

$$G_m i\lambda\rho\, J_m i\lambda\rho' = (i^m G_m i\lambda\rho)(i^{-m} J_m i\lambda\rho'),$$

the product of two factors constantly positive, so that the modulus of the elementary term

$$\cos\lambda(z - z') G_m i\lambda\rho\, J_m i\lambda\rho' \cos m(\phi - \phi')$$

is not greater than $G_m i\lambda\rho\, J_m i\lambda\rho'$.

But $\int_0^\infty d\lambda \sum_m' G_m i\lambda\rho \, J_m i\lambda\rho'$

$= \int_0^\infty d\lambda \, . \, G_0 i\lambda(\rho - \rho')$, which is finite, so that the sum-integral converges absolutely,

The same applies to the sum-integral that has been added to T to obtain V.

Hence $V = 4/\pi \sum_m' \cos m(\phi - \phi')$.

$$\int_0^\infty \cos\lambda(z - z') \frac{J_m i\lambda\rho'}{J_m i\lambda a} (J_m i\lambda a \, G_m i\lambda\rho - J_m i\lambda\rho \, G_m i\lambda a) d\lambda.$$

The integrand here is a *uniform, even,* function of λ, so that the integral may be written

$$\frac{1}{2i} \int e^{-\lambda(z-z')} \frac{J_m\lambda\rho'}{J_m\lambda a} (J_m\lambda a \, G_m\lambda\rho - J_m\lambda\rho \, G_m\lambda a) d\lambda \Big| \text{SN}.$$

If $z > z'$ and $\rho > \rho'$, the integrand vanishes at infinity in the eastern half of the λ plane. The infinities are simple poles at the zeros of $J_m\lambda a$. The integral therefore equals $-2\pi i$ (residues to right of O), and

$$V = 4/a \sum_m' \cos m(\phi - \phi') \sum_\lambda e^{-\lambda(z-z')} J_m\lambda\rho \, J_m\lambda\rho' \, G_m\lambda a / J'_m\lambda a$$

$$= 4/a^2 \sum_m' \cos m(\phi - \phi') \sum_\lambda e^{-\lambda(z-z')} J_m\lambda\rho \, J_m\lambda\rho' / \overline{\lambda(J'_m\lambda a)^2},$$

using the relation $G_m\lambda a \, J'_m\lambda a - J_m\lambda a \, G'_m\lambda a = 1/\lambda a$.

Here the λ's are the positive zeros, in ascending order, of

$$J_m\lambda a, \text{ and } z > z'.$$

(3) The ϕ form for V is deduced from the form (1) by a process similar to that used in proving 17 (3).

Take the complex integral

$$\int \frac{\cos m(\pi - \phi + \phi')}{\sin m\pi} \frac{J_m i\lambda\rho'}{J_m i\lambda a} (J_m i\lambda a \, G_m i\lambda\rho - J_m i\lambda\rho \, G_m i\lambda a) dm$$

along the two paths from S to N used in 17.

For a very large m, the factors of the integrand involving the Bessels approximate to

$$\left(\frac{\rho'}{a}\right)^m \frac{\pi}{2\sin m\pi}\left\{\left(\frac{a}{\rho}\right)^m - \left(\frac{\rho}{a}\right)^m\right\}\frac{1}{\Pi m \,.\, \Pi(-m)}$$
$$= \frac{1}{2m}\left\{\left(\frac{\rho'}{\rho}\right)^m - \left(\frac{\rho\rho'}{a^2}\right)^m\right\},$$

which vanishes on the eastern side of the plane if $\rho > \rho'$.
The other factor vanishes if $0 < \phi - \phi' < 2\pi$.
Under these conditions, the sum of the two integrals is

$$\frac{2}{\pi}(-2\pi i)\sum_m{}' \frac{J_m i\lambda\rho'}{J_m i\lambda a}(J_m i\lambda a\, G_m i\lambda\rho - J_m i\lambda\rho\, G_m i\lambda a)\cos m(\phi - \phi').$$

But that sum is

$$2\int \frac{\cos m(\pi - \phi + \phi')}{\sin m\pi}\left(\frac{J_m i\lambda\rho'}{J_m i\lambda a} - \frac{J_{-m} i\lambda\rho'}{J_{-m} i\lambda a}\right)$$
$$[J_m i\lambda a\, G_m i\lambda\rho - J_m i\lambda\rho\, G_m i\lambda a]dm \;\Big|\; \text{ON},$$

the function in square brackets being *even* in m.
Hence, from (1) and using the definition of G,

$$V = \frac{4}{\pi^2}\int_0^\infty \cos\lambda(z - z')d\lambda$$
$$\int_0^\infty \left\{\begin{array}{l}\cosh s(\pi - \phi + \phi')(J_{is} i\lambda a\, G_{is} i\lambda\rho' - J_{is} i\lambda\rho'\, G_{is} i\lambda a). \\ \quad (J_{is} i\lambda a\, G_{is} i\lambda\rho - J_{is} i\lambda\rho\, G_{is} i\lambda a)ds/(J_{is} i\lambda a\, J_{-is} i\lambda a)\end{array}\right\}$$
$$0 < \phi - \phi' < 2\pi.$$

21. *Space bounded by two axial planes*

$$\phi = 0,\ \phi = \alpha > 0.$$

(1) Starting with the ϕ form for T, 18 (3),
and observing that the factor of the integrand involving ϕ is

$$\cosh s(\pi - \phi + \phi') \quad \text{when} \quad \phi > \phi',$$
$$\text{but} \quad \cosh s(\pi - \phi' + \phi) \quad \text{when} \quad \phi < \phi',$$

we proceed as in 19 (1), and obtain

$$V = \frac{8}{\pi^2}\int_0^\infty \cos\lambda(z - z')d\lambda.$$
$$\int_0^\infty e^{-s\pi}(\sinh s\pi/\sinh s\alpha)\sinh s(\alpha - \phi)\sinh s\phi'\, G_{is} i\lambda\rho\, G_{is} i\lambda\rho' ds.$$
$$\phi > \phi'.$$

(2) Bearing in mind that $i^s G_s$ is an even function of s, we write the second integral

$$\int (\sin s\pi/\sin s\alpha)\sin s(\alpha-\phi)\sin s\phi'(i^s G_s i\lambda\rho)(i^s G_s i\lambda\rho')ids \;\Big|\; \text{ON}$$

$$= -\frac{\pi i}{2}\int \frac{\sin s(\alpha-\phi)\sin s\phi'}{\sin s\alpha} G_s i\lambda\rho\, J_s i\lambda\rho' ds \;\Big|\; \text{SN}$$

$$= (-2\pi i)\ (\text{residues to right of O}),\ \text{if}\ \rho>\rho'.$$

Hence

$$V = \frac{8}{\alpha}\int_0^\infty \cos\lambda(z-z')d\lambda \sum_m \sin\frac{m\pi\phi}{\alpha}\sin\frac{m\pi\phi'}{\alpha} G_{\frac{m\pi}{\alpha}} i\lambda\rho\, J_{\frac{m\pi}{\alpha}} i\lambda\rho'.$$

$$\rho>\rho'.$$

(3) The sum-integral just written converges absolutely, and we may put (writing s for $m\pi/\alpha$)

$$V = \frac{8}{\alpha}\sum_m \sin s\phi\,.\,\sin s\phi' \int_0^\infty \cos\lambda(z-z')G_s i\lambda\rho\, J_s i\lambda\rho' d\lambda.$$

If $z>z'$, $\rho>\rho'$, the integral is

$$\frac{1}{2i}\int \{e^{\lambda(z-z')} + e^{-\lambda(z-z')}\} G_s\lambda\rho\,.\,J_s\lambda\rho' d\lambda \;\Big|\; \text{ON}$$

$$= \frac{1}{2i}\int e^{\lambda(z-z')} G_s\lambda\rho\, J_s\lambda\rho' d\lambda \;\Big|\; \text{OW}$$

$$+\frac{1}{2i}\int e^{-\lambda(z-z')} G_s\lambda\rho\, J_s\lambda\rho' d\lambda \;\Big|\; \text{OE}$$

$$= \frac{1}{2i}\int e^{-\lambda(z-z')} J_s\lambda\rho' \{G_s\lambda\rho - e^{is\pi} G_s(-\lambda\rho)\} d\lambda \;\Big|\; \text{OE}$$

$$= \frac{\pi}{2}\int_0^\infty e^{-\lambda(z-z')} J_s\lambda\rho\, J_s\lambda\rho' d\lambda. \quad (\S 1.)$$

Substitution in the form (2) gives

$$V = \frac{4\pi}{\alpha}\sum_m \sin\frac{m\pi\phi}{\alpha}\sin\frac{m\pi\phi'}{\alpha}\int_0^\infty e^{-\lambda(z-z')} J_{\frac{m\pi}{\alpha}}\lambda\rho\,.\,J_{\frac{m\pi}{\alpha}}\lambda\rho'.\,d\lambda.$$

$$z>z'.$$

22. *Space bounded externally by two parallel planes, and a cylinder,* $z=0, \quad z=c, \quad \rho=a.$

(1) From 19 (2)

$$V=\frac{8}{c}\sum_p \sin\lambda z\,.\,\sin\lambda z'.$$
$$\sum_m{}' \frac{J_m i\lambda\rho'}{J_m i\lambda a}(J_m i\lambda a\, G_m i\lambda\rho - J_m i\lambda\rho\, G_m i\lambda a)\cos m(\phi-\phi').$$
$$\lambda=p\pi/c, \quad \rho>\rho'.$$

(2) From 20 (2) by the same work as in 19 (1),

$$V=\frac{8}{a^2}\sum_m{}'\cos m(\phi-\phi')\sum_\lambda \frac{\sinh\lambda(c-z)\sinh\lambda z'}{\sinh\lambda c} J_m\lambda\rho\,.\,J_m\lambda\rho'/\overline{\lambda(J'_m\lambda a)^2}.$$

The λ's are the positive zeros of $J_m\lambda a$, $z>z'$.

(3) From (1), utilizing the work of 20 (3), we write down at once

$$V=\frac{8}{\pi c}\sum_p \sin\lambda z\,.\,\sin\lambda z'.$$
$$\int_0^\infty \cosh s(\pi-\phi+\phi')f(\rho)\,.\,f(\rho')ds/(J_{is}i\lambda a\,.\,J_{-is}i\lambda a),$$

where
$$f(\rho)=J_{is}i\lambda a\, G_{is}i\lambda\rho - J_{is}i\lambda\rho\, G_{is}i\lambda a,$$
$$\lambda=p\pi/c, \quad 0<\phi-\phi'<2\pi.$$

23. *Space bounded by two parallel planes, and two axial planes* $z=0, \quad z=c, \quad \phi=0, \quad \phi=\alpha.$

(1) Starting from 19 (3), the process of 21 (1) gives

$$V=\frac{16}{\pi c}\sum_p \sin\lambda z\,.\,\sin\lambda z'.$$
$$\int_0^\infty e^{-s\pi}(\sinh s\pi/\sinh s\alpha)\sinh s(\alpha-\phi)\sinh s\phi'.\, G_{is}i\lambda\rho\,.\,G_{is}i\lambda\rho'.\,ds$$
$$\lambda=p\pi/c, \quad \phi>\phi',$$

(2) From 21 (3),

$$V=\frac{8\pi}{\alpha}\sum_m \sin s\phi\,\sin s\phi'\int_0^\infty \frac{\sinh\lambda(c-z)\sinh\lambda z'}{\sinh\lambda c} J_s\lambda\rho\, J_s\lambda\rho' d\lambda.$$
$$s=m\pi/\alpha, \quad z>z'.$$

(3) From (1) as in 21 (2),

$$V=\frac{16\pi}{c\alpha}\sum_p \sin\lambda z\sin\lambda z'.$$
$$\sum_m \sin s\phi\,\sin s\phi'\, G_s i\lambda\rho\, J_s i\lambda\rho'.$$
$$\lambda=p\pi/c, \quad s=m\pi/\alpha, \quad \rho>\rho'.$$

24. *Space bounded by two axial planes and a cylinder*

$$\phi=0, \quad \phi=a, \quad \rho=a.$$

(1) From 21 (2),

$$V=\frac{8}{a}\int_0^\infty \cos\lambda(z-z')d\lambda.$$
$$\sum_m \sin s\phi \sin s\phi'(J_s i\lambda\rho'/J_s i\lambda a)(J_s i\lambda a\, G_s i\lambda\rho - J_s i\lambda\rho\, G_s i\lambda a),$$
$$s=m\pi/a, \quad \rho>\rho'.$$

(2) From 20 (3),

$$V=\frac{8}{\pi^2}\int_0^\infty \cos\lambda(z-z')d\lambda.$$
$$\int_0^\infty \frac{\sinh s\pi}{\sinh sa}\sinh s(a-\phi)\sinh s\phi'.f(\rho)f(\rho')\cdot\frac{ds}{J_{is}i\lambda a\, J_{-is}i\lambda a}.$$
$$f(\rho) \text{ as in 22 (3)}, \quad \phi>\phi'.$$

(3) From (1) as in 20 (2),

$$V=\frac{8\pi}{a^2 a}\sum_m \sin s\phi \sin s\phi'.$$
$$\sum_\lambda e^{-\lambda(z-z')}J_s\lambda\rho\,.\,J_s\lambda\rho'/\overline{\lambda(J'_s\lambda a)^2}.$$
$$s=m\pi/a, \ \lambda \text{ a positive zero of } J_s\lambda a, \ z>z'.$$

25. *Space bounded by two axial planes, two parallel planes, and a cylinder*

$$\phi=0, \quad \phi=a, \quad z=0, \quad z=c, \quad \rho=a.$$

(1) From 24 (3)

$$V=\frac{16\pi}{a^2 a}\sum_m \sin s\phi \sin s\phi'$$
$$\sum_\lambda \frac{\sinh\lambda(c-z)\sinh\lambda z'}{\sinh\lambda c}J_s\lambda\rho\,.\,J_s\lambda\rho'/\overline{\lambda(J'_s\lambda a)^2}.$$
$$s=m\pi/a, \ \lambda \text{ a positive zero of } J_s\lambda a, \ z>z'.$$

(2) From 23 (3),

$$V=\frac{16\pi}{ca}\sum_p \sin\lambda z \sin\lambda z'.$$
$$\sum_m \sin s\phi \sin s\phi'(J_s i\lambda\rho'/J_s i\lambda a)(J_s i\lambda a\, G_s i\lambda\rho - J_s i\lambda\rho\, G_s i\lambda a).$$
$$\lambda=p\pi/c, \quad s=m\pi/a, \quad \rho>\rho'.$$

(3) In 22 (3), instead of $\cosh s(\pi-\phi+\phi')$,
write $2(\sinh s\pi/\sinh sa)\sinh s(a-\phi)\sinh s\phi'$.
$$\phi>\phi'.$$

26. *Space bounded by two parallel planes, two axial planes, and two cylinders*

$$z=0,\ z=c,\ \phi=0,\ \phi=\alpha,\ \rho=a,\ \rho=b,\ b>a.$$

Since this is the first occurrence of a boundary partly composed of *two* cylinders, we can deduce from preceding results only one of the forms for V. From this the other two are derived directly.

(1) Taking the function of 23 (3) we subtract from it a potential taking the same value on the cylinders and vanishing on the planes.

This potential is clearly obtained by writing for $G_s i\lambda\rho\, J_s i\lambda\rho'$ in 23 (3),

$$G_s i\lambda b \,.\, J_s i\lambda\rho' \,.\, \frac{J_s i\lambda a\, G_s i\lambda\rho - J_s i\lambda\rho\, G_s i\lambda a}{J_s i\lambda a\, G_s i\lambda b - J_s i\lambda b\, G_s i\lambda a}$$

$$+ J_s i\lambda a\ \ G_s i\lambda\rho' \,.\, \frac{J_s i\lambda\rho\, G_s i\lambda b - J_s i\lambda b\, G_s i\lambda\rho}{J_s i\lambda a\, G_s i\lambda b - J_s i\lambda b\, G_s i\lambda a},$$

for this expression is

$$G_s i\lambda b\, J_s i\lambda\rho' \text{ for } \rho=b, \text{ and } J_s i\lambda a\, G_s i\lambda\rho' \text{ for } \rho=a.$$

A slight reduction gives

$$V=\frac{16\pi}{c\alpha}\sum_p \sin\lambda z \sin\lambda z' \sum_m \sin s\phi \sin s\phi'.$$
$$(J_s i\lambda a\, G_s i\lambda\rho' - J_s i\lambda\rho'\, G_s i\lambda a)(J_s i\lambda b\, G_s i\lambda\rho - J_s i\lambda\rho\, G_s i\lambda b)$$
$$\div (J_s i\lambda a\, G_s i\lambda b - J_s i\lambda b\, G_s i\lambda a).$$
$$\lambda=p\pi/c,\ s=m\pi/\alpha,\ \rho>\rho'.$$

(2) To deduce the z form from this, suppose $z>z'$, change the order of summation, and consider the function of λ

$$\frac{\sinh\lambda(c-z)\sinh\lambda z'}{\sinh\lambda c}\cdot\frac{(J_s\lambda a G_s\lambda\rho' - J_s\lambda\rho' G_s\lambda a)(J_s\lambda b G_s\lambda\rho - J_s\lambda\rho G_s\lambda b)}{J_s\lambda a\, G_s\lambda b - J_s\lambda b\, G_s\lambda a}.$$

This is a uniform, odd function of λ which, if $z>z'$, $\rho>\rho'$, vanishes for every infinite λ. Hence the total sum of the residues of the function vanishes. The infinities of the function are simple poles, namely, the (pure imaginary) zeros of $\sinh\lambda c$, and the (real) zeros of $J_s\lambda a\, G_s\lambda b - J_s\lambda b\, G_s\lambda a$.

Substituting one series of residues for the other in (1) gives

$$V = -\frac{16\pi}{a} \sum_m \sin s\phi \sin s\phi' . \sum_\lambda \frac{\sinh \lambda(c-z)\sinh \lambda z'}{\sinh \lambda c} .$$
$$(J_s\lambda a\, G_s\lambda\rho' - J_s\lambda\rho'\, G_s\lambda a)(J_s\lambda b\, G_s\lambda\rho - J_s\lambda\rho\, G_s\lambda b)$$
$$\div \frac{d}{d\lambda}(J_s\lambda a\, G_s\lambda b - J_s\lambda b\, G_s\lambda a).$$

$s = m\pi/a$, the λ's are the positive zeros in order of

$$J_s\lambda a\, G_s\lambda b - J_s\lambda b\, G_s\lambda a, \quad z > z'.$$

(3) By a similar process, (1) yields the ϕ form also.

Consider the function of s

$$\frac{\sin s(\alpha-\phi)\sin s\phi'}{\sin s\alpha} \frac{(J_s i\lambda a\, G_s i\lambda\rho' - J_s i\lambda\rho'\, G_s i\lambda a)(J_s i\lambda b\, G_s i\lambda\rho - J_s i\lambda\rho\, G_s i\lambda b)}{J_s i\lambda a\, G_s i\lambda b - J_s i\lambda b\, G_s i\lambda a}.$$

This is a uniform, odd function of s, vanishing for every infinite s if $\phi > \phi'$, $\rho > \rho'$. The infinities are simple poles at the (real) zeros of $\sin s\alpha$, and the (pure imaginary) zeros of $J_s i\lambda a\, G_s i\lambda b - J_s i\lambda b\, G_s i\lambda a$. Hence, as before,

$$V = -\frac{16\pi}{c} \sum_p \sin\lambda z \sin\lambda z' \sum_s \frac{\sinh s(\alpha-\phi)\sinh s\phi'}{\sinh s\alpha} .$$
$$(J_{is} i\lambda a\, G_{is} i\lambda\rho' - J_{is} i\lambda\rho'\, G_{is} i\lambda a)(J_{is} i\lambda b\, G_{is} i\lambda\rho - J_{is} i\lambda\rho\, G_{is} i\lambda b)$$
$$\div \frac{d}{ds}(J_{is} i\lambda a\, G_{is} i\lambda b - J_{is} i\lambda b\, G_{is} i\lambda a).$$

$\lambda = p\pi/c$, the s's are the positive zeros of

$$J_{is} i\lambda a\, G_{is} i\lambda b - J_{is} i\lambda b\, G_{is} i\lambda a, \quad \phi > \phi'.$$

In the forms (2) and (3) it should be noticed that the expression for V is unaltered when ρ, ρ' are interchanged, on account of the equation defining λ or s.

These forms show very strikingly how the Bessels of *real* factor and rank bear the same sort of relation to the parallel planes as the functions of *pure imaginary* factor and rank bear to the axial planes.

From the solutions of this paragraph all those that precede may, of course, be deduced, but the reduction is not always easy, nor its validity obvious.

Spherical Harmonics.

The spherical polar coordinates of the variable point are r, θ, ϕ; of the pole r', θ', ϕ'.

We write $r = ae^{\rho}$, $r' = ae^{\rho'}$, so that ρ runs from $-\infty$ to $+\infty$ as r runs from 0 to ∞, and $\rho = 0$ when $r = a$.

Also, as in 16, we write $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$, and, as usual, $\cos\theta = \mu$, $\cos\theta' = \mu'$.

27. *Whole of space.*

$$T = 1/\sqrt{(r^2 - 2rr'\cos\gamma + r'^2)}, \text{ which if } r > r'$$

$$= \frac{1}{r}\left\{P_0(\cos\gamma) + \frac{r'}{r}P_1(\cos\gamma) + \left(\frac{r'}{r}\right)^2 P_2(\cos\gamma) + , \text{ etc.}\right\}$$

Multiplying by $\sqrt{rr'}$, this may be written

$$T\sqrt{rr'} = \sum_n e^{-(n+\frac{1}{2})(\rho-\rho')}P_n(\cos\gamma)$$

$$= \sum_n (-)^n e^{-(n+\frac{1}{2})(\rho-\rho')}P_n(-\cos\gamma).$$

The point $n = -\frac{1}{2}$ in the plane of n we shall always denote by C; the path SCN is the straight path through C from

$$-\tfrac{1}{2} - \infty i \text{ to } -\tfrac{1}{2} + \infty i.$$

The approximate form of $P_n(-\cos\gamma)$ for a very large n is

$$\sqrt{\frac{2}{n\pi\sin\theta}} \cdot \cos\left\{(n+\tfrac{1}{2})(\pi-\gamma) - \frac{\pi}{4}\right\}. \quad (\S 14.)$$

Hence

$$\frac{\pi}{2\pi i}\int \frac{e^{-(n+\frac{1}{2})(\rho-\rho')}P_n(-\cos\gamma)}{\cos(n+\frac{1}{2})\pi} dn \;\Big|\; \text{SCN}$$

$$= (-2\pi i) \text{ (residues to right of C), if } \rho > \rho',$$

$$= T\sqrt{rr'}, \text{ or putting } n + \tfrac{1}{2} = i\lambda$$

$$T\sqrt{rr'} = \int_0^\infty \frac{\cos\lambda(\rho-\rho')P_{-\frac{1}{2}+i\lambda}(-\cos\gamma)d\lambda}{\cosh\lambda\pi}$$

(Heine II. p. 219.

$$= \frac{1}{\pi}\int_0^\infty \cos\lambda(\rho-\rho')S_{-\frac{1}{2}+i\lambda}(\cos\gamma)d\lambda.$$

In expanded form

(1) $$\mathrm{T}\sqrt{rr'} = 2\sum_n e^{-(n+\frac{1}{2})(\rho-\rho')}\cdot \sum_m{}' (-)^m \frac{\Pi(m+n)\Pi(m-n-1)}{\Pi n\,.\,\Pi(-n-1)} \mathrm{P}_n^m\mu\,.\,\mathrm{P}_n^m\mu' \cos m(\phi-\phi'),$$

the m summation stopping, of course, at $m=n$, $r>r'$.

(2) $$\mathrm{T}\sqrt{rr'} = \frac{2}{\pi}\int_0^\infty \cos\lambda(\rho-\rho')d\lambda \sum_m{}' \mathrm{S}_n^m\mu\,.\,\mathrm{P}_n^m\mu'\,.\cos m(\phi-\phi').$$

$$n=-1/2+i\lambda,\ \theta>\theta'.$$

(3) $$\mathrm{T}\sqrt{rr'} = \frac{i}{\pi}\int_0^\infty \cos\lambda(\rho-\rho')d\lambda.$$

$$\int_0^\infty \frac{\cosh s(\pi-\phi+\phi')}{\sinh s\pi}(\mathrm{S}_n^{iu}\mu\,\mathrm{P}_n^{iu}\mu' - \mathrm{S}_n^{-iu}\mu\,\mathrm{P}_n^{-iu}\mu')ds$$

from 18 (4). $n=-1/2+i\lambda$, $0<\phi-\phi'<2\pi$.

28. *Space bounded by two spheres*

$$\rho=0,\ \rho=c.$$

(1) From $\mathrm{T}\sqrt{rr'} = \sum_n e^{-(n+\frac{1}{2})(\rho-\rho')}\mathrm{P}_n\cos\gamma$, $\rho>\rho'$, precisely as in 19,

$$\mathrm{V}\sqrt{rr'} = 2\sum_n \frac{\sinh(n+\frac{1}{2})(c-\rho)\sinh(n+\frac{1}{2})\rho'}{\sinh(n+\frac{1}{2})c}\,\mathrm{P}_n\cos\gamma.$$

(2) The function of n (odd in $n+\frac{1}{2}$),

$$\frac{\sinh(n+\frac{1}{2})(c-\rho)\sinh(n+\frac{1}{2})\rho'}{\sinh(n+\frac{1}{2})c\,.\,\cos(n+\frac{1}{2})\pi}\,\mathrm{P}_n(-\cos\gamma),$$

where $\rho>\rho'$, $\gamma>0$, vanishes for every infinite n, and the total sum of its residues is *nil*, that is,

$$-\frac{2}{\pi}\sum_n \frac{\sinh(n+\frac{1}{2})(c-\rho)\sinh(n+\frac{1}{2})\rho'}{\sinh(n+\frac{1}{2})c}(-)^n\mathrm{P}_n(-\cos\gamma)$$

$$+\frac{2}{c}\sum_p \frac{\sin\lambda\rho\sin\lambda\rho'}{\cosh\lambda\pi}\,\mathrm{P}_{-\frac{1}{2}+i\lambda}(-\cos\gamma)=0,$$

where $\lambda=p\pi/c$.

Therefore from (1)

$$V\sqrt{rr'} = \frac{2\pi}{c} \sum_p \frac{\sin\lambda\rho \sin\lambda\rho'}{\cosh \lambda\pi} P_{-\frac{1}{2}+i\lambda}(-\cos\gamma),$$

$$\text{or } V\sqrt{rr'} = \frac{4}{c} \sum_p \sin\lambda\rho \sin\lambda\rho' \sum_m' S_n^m\mu \, P_n^m\mu' \cdot \cos m(\phi - \phi').$$

$$\lambda = p\pi/c, \quad n = -1/2 + i\lambda, \quad \theta > \theta'.$$

If c is a small fraction, that is, if the thickness of the spherical shell is small in comparison with its radius, the first term of the unexpanded form for $V\sqrt{rr'}$ will be a good approximation, γ not being a very small angle.

The limiting form, as c diminishes more and more, is

$$V\sqrt{rr'} = \frac{4}{\sqrt{2c\sin\gamma}} e^{-\pi\gamma/c} \sin\pi\rho/c \cdot \sin\pi\rho'/c,$$

an extension of the result of 19.

(3) Substituting the ϕ form of $P(-\cos\gamma)$ in (2), we get

$$V\sqrt{rr'} = 2i/c \sum_p \sin\lambda\rho \sin\lambda\rho'$$

$$\int_0^\infty \frac{\cosh s(\pi - \phi + \phi')}{\sinh s\pi} (S_n^{is}\mu \, P_n^{is}\mu' - S_n^{-is}\mu \, P_n^{-is}\mu') ds$$

$$\lambda = p\pi/c, \quad n = -1/2 + i\lambda, \quad 0 < \phi - \phi' < 2\pi.$$

29. *Space bounded by a single cone*

$$\cos\theta = \beta.$$

(1) From the θ form of T

$$V\sqrt{rr'} = \frac{2}{\pi} \int_0^\infty \cos\lambda(\rho - \rho') d\lambda$$

$$\sum_m' \frac{P_n^m\mu'}{P_n^m\beta} (P_n^m\beta \, S_n^m\mu - P_n^m\mu S_n^m\beta) \cos m(\phi - \phi').$$

$$n = -1/2 + i\lambda, \quad \theta > \theta'.$$

(2) In (1) we may change the order of summation, integration. (Cf. 20.) Then

$$\frac{2}{\pi} \int_0^\infty \cos\lambda(\rho - \rho') \frac{P\mu'}{P\beta} (P\beta \, S\mu - P\mu S\beta) d\lambda$$

$$= \frac{1}{\pi i} \int e^{-(n+\frac{1}{2})(\rho - \rho')} \frac{P_n^m\mu'}{P_n^m\beta} (P_n^m\beta \, S_n^m\mu - P_n^m\mu \, S_n^m\beta) dn \;\Big|\; \text{SCN}$$

$$= (-2\pi i) \text{ (residues to right of C), if } \theta > \theta', \; \rho > \rho'.$$

The poles of the integrand are the (real) zeros of $P_n^m\beta$.

Hence

$$V\sqrt{rr'} = 2\sum_m{}'\cos m(\phi-\phi')\sum_n e^{-(n+\frac{1}{2})(\rho-\rho')}P_n^m\mu.P_n^m\mu'S_n^m\beta\Big/\frac{d}{dn}P_n^m\beta.$$

The n's are the zeros $> -\frac{1}{2}$ of $P_n^m\beta$, and $r>r'$.

(3) It has been shown in §12, 2 that

$$P_n^m\mu\,S_n^m\mu' - P_n^m\mu'\,S_n^m\mu \text{ is an even function of } m.$$

Hence from (1), by a process quite similar to that of 20 (3), and using the equation of 12 (2) we find

$$V\sqrt{rr'}$$
$$=\frac{1}{\pi^2}\int_0^\infty\cos\lambda(\rho-\rho')d\lambda\int_0^\infty\cosh s(\pi-\phi+\phi')f(\theta)f(\theta')ds/(P_n^{is}\beta.P_n^{-is}\beta)$$

where $f(\theta)=P_n^{is}\beta\,.\,S_n^{is}\mu - P_n^{is}\mu\,S_n^{is}\beta,$

$$n=-1/2+i\lambda,\ 0<\phi-\phi'<2\pi.$$

30. *Space bounded by two axial planes*

$$\phi=0,\ \phi=\alpha.$$

(1) From 27 (3), as in 21 (1),

$$V\sqrt{rr'}=\frac{2i}{\pi}\int_0^\infty\cos\lambda(\rho-\rho')d\lambda.$$
$$\int_0^\infty\frac{\sinh s(\alpha-\phi)\sinh s\phi'}{\sinh s\alpha}(S_n^{is}\mu\,P_n^{is}\mu' - S_n^{-is}\mu\,P_n^{-is}\mu')ds.$$
$$n=-1/2+i\lambda,\ \phi>\phi'.$$

(2) The second integral can be expressed as

$$-\int\frac{\sin m(\alpha-\phi)\sin m\phi'}{\sin m\alpha}S_n^m\mu\,.\,P_n^m\mu'.\,dm\ \Big|\ \text{SON}$$

which if $\phi>\phi'$, $\theta>\theta'$ equals $2\pi i$ (residues of integrand to right of O). Thus

$$V\sqrt{rr'}=\frac{4}{\alpha}\int_0^\infty\cos\lambda(\rho-\rho')d\lambda\sum_m\sin s\phi\sin s\phi'\,S_n^s\mu\,P_n^s\mu'.$$
$$n=-1/2+i\lambda,\ s=m\pi/\alpha,\ \theta>\theta'.$$

(3) We may write (2)

$$V\sqrt{rr'}=\frac{4}{\alpha}\sum_m\sin s\phi\sin s\phi'\int_0^\infty\cos\lambda(\rho-\rho')S_n^s\mu\,P_n^s\mu'd\lambda.$$

The integral is equivalent to

$$\frac{1}{2i}\int e^{-(n+\frac{1}{2})(\rho-\rho')}\Pi(s+n)\Pi(s-n-1)\mathrm{P}_n^s(-\mu)\mathrm{P}_n^s\mu'.\,dn \;\Big|\; \mathrm{SCN}.$$

The integrand vanishes at infinity to the right of C, provided

$$\rho>\rho',\ \theta>\theta'.$$

Since s is a real positive, the infinities to the right of C are the poles of $\Pi(s-n-1)$; for these,

$$n=s+p,\ p \text{ zero or a positive integer.}$$

Now $\Pi(-p-1)=-\pi/(\sin p\pi\,.\,\Pi p)$, so that the residue of $\Pi(-p-1)$ when p is an integer is $(-)^{p-1}/\Pi p$.
Also when p is an integer, $\mathrm{P}_{s+p}^s(-\mu)=(-)^p\mathrm{P}_{s+p}^s\mu$.
Hence the above integral equals

$$\pi\sum_p e^{-(s+p+\frac{1}{2})(\rho-\rho')}(\Pi\overline{2s+p}/\Pi p)\mathrm{P}_{s+p}^s\mu\,.\,\mathrm{P}_{s+p}^s\mu',$$

and

$$\mathrm{V}\sqrt{rr'}=\frac{4\pi}{a}\sum_m \sin s\phi\sin s\phi'.$$
$$\sum_p e^{-(s+p+\frac{1}{2})(\rho-\rho')}(\Pi\overline{2s+p}/\Pi p)\mathrm{P}_{s+p}^s\mu\,.\,\mathrm{P}_{s+p}^s\mu',$$
$$s=m\pi/a,\ r>r'.$$

Cf. Lord Kelvin and Tait, *Nat. Phil*, Vol. I. App. B. (m).

31. *Space bounded by two spheres and a cone*

$$\rho=a,\ \rho=c,\ \cos\theta=\beta.$$

(1) From 28 (2),

$$\mathrm{V}\sqrt{rr'}=\frac{4}{c}\sum_p \sin\lambda\rho.\sin\lambda\rho'\,\Sigma'_m\frac{\mathrm{P}_n^m\mu'}{\mathrm{P}_n^m\beta}(\mathrm{P}_n^m\beta\,\mathrm{S}_n^m\mu-\mathrm{P}_n^m\mu\,\mathrm{S}_n^m\beta)\cos m(\phi-\phi').$$

$$\lambda=p\pi/c,\ n=-1/2+i\lambda,\ \theta>\theta'.$$

(2) From 29 (2),

$$\mathrm{V}\sqrt{rr'}=4\,\Sigma'_m\cos m(\phi-\phi').$$
$$\sum_n\frac{\sinh(n+\frac{1}{2})(c-\rho)\sinh(n+\frac{1}{2})\rho'}{\sinh(n+\frac{1}{2})c}(\mathrm{P}_n^m\mu\,\mathrm{P}_n^m\mu'\,\mathrm{S}_n^m\beta\Big/\frac{d}{dn}\mathrm{P}_n^m\beta.$$

The n's are the zeros $>-\frac{1}{2}$ of $\mathrm{P}_n^m\beta$; $r>r'$.

(3) As, in 29, (3) is obtained from (1), so here from (1)

$$V\sqrt{rr'}=\frac{2}{\pi c}\sum_p \sin\lambda\rho\sin\lambda\rho'\int_0^\infty \cosh s(\pi-\phi+\phi')f(\theta)f(\theta')ds/(P_n^{iu}\beta\, P_n^{-iu}\beta)$$

$$\lambda=p\pi/c,\ n=-1/2+i\lambda,\ f(\theta)=P_n^{iu}\beta\, S_n^{iu}\mu-P_n^{iu}\mu\, S_n^{iu}\beta,$$

$$0<\phi-\phi'<2\pi.$$

32. *Space bounded by two spheres and two axial planes*

$$\rho=0,\ \rho=c,\ \phi=0,\ \phi=\alpha.$$

(1) From 28 (3),

$$V\sqrt{rr'}=\frac{4i}{c}\sum_p \sin\lambda\rho\,\sin\lambda\rho'.$$

$$\int_0^\infty \frac{\sinh s(\alpha-\phi)\sinh s\phi'}{\sinh s\alpha}(S_n^{iu}\mu\, P_n^{iu}\mu'-S_n^{-iu}\mu\, P_n^{-iu}\mu')ds$$

$$\lambda=p\pi/c,\ n=-1/2+i\lambda,\ 0<\phi-\phi'<2\pi.$$

(2) From 30 (3),

$$V\sqrt{rr'}=\frac{8\pi}{\alpha}\sum_m \sin s\phi\,.\,\sin s\phi'.$$

$$\sum_p(\Pi\overline{2s+p}/\Pi p)\frac{\sinh(s+p+\frac{1}{2})(c-\rho)}{\sinh(s+p+\frac{1}{2})c}\frac{\sinh(s+p+\frac{1}{2})\rho'}{}P^s_{s+p}\mu\,.\,P^s_{s+p}\mu'.$$

$$s=m\pi/\alpha,\ r>r'.$$

(3) From (1) as in 30 (2),

$$V\sqrt{rr'}=\frac{8\pi}{c\alpha}\sum_p \sin\lambda\rho\,\sin\lambda\rho',$$

$$\sum_m \sin s\phi\,\sin s\phi'.\,S_n^s\mu\,P_n^s\mu',$$

$$s=m\pi/\alpha,\ n=-1/2+i\lambda,\ \lambda=p\pi/c,\ \theta>\theta'.$$

33. *Space bounded by two axial planes and cone*

$$\phi=0,\ \phi=\alpha,\ \cos\theta=\beta.$$

(1) From 30 (2),

$$V\sqrt{rr'}=\frac{4}{\alpha}\int_0^\infty \cos\lambda(\rho-\rho')d\lambda.$$

$$\sum_m \sin s\phi\,.\,\sin s\phi'\,.\,\frac{P_n^m\mu'}{P_n^m\beta}(P_n^m\beta\, S_n^m\mu-P_n^m\mu\, S_n^m\beta).$$

$$s=m\pi/\alpha,\ n=-1/2+i\lambda,\ \theta>\theta'.$$

(2) In 29 (3) replace $\cosh s(\pi-\phi+\phi')$ by

$$2\sinh s(\alpha-\phi)\sinh s\phi'.\ \sinh s\pi/\sinh s\alpha,$$

$$\phi>\phi'.$$

(3) From (1) as in 29 (2),

$$\mathrm{V}\sqrt{rr'}=\frac{4\pi}{\alpha}\sum_m \sin s\phi\,.\,\sin s\phi'.$$

$$\sum_n e^{-(n+\frac{1}{2})(\rho-\rho')}\mathrm{P}^s_{n\mu}\,\mathrm{P}^s_{n\mu'}\Big/\frac{d}{dn}\,\mathrm{P}^s_{n\beta}.$$

$$s=m\pi/\alpha,\ n \text{ a zero } >-\tfrac{1}{2} \text{ of } \mathrm{P}^s_{n\beta},\ r>r'.$$

34. *Space bounded by two spheres, two axial planes, and a cone*

$$\rho=0,\ \rho=c,\ \phi=0,\ \phi=\alpha,\ \cos\theta=\beta.$$

(1) In 33 (3) change $e^{-(n+\frac{1}{2})(\rho-\rho')}$ into

$$2\sinh(n+\tfrac{1}{2})(c-\rho)\sinh(n+\tfrac{1}{2})\rho'/\sinh(n+\tfrac{1}{2})c.$$

$$r>r'.$$

(2) In 32 (3) change $\mathrm{S}^s_{n\mu}\,\mathrm{P}^s_{n\mu'}$ into

$$(\mathrm{P}^s_{n\mu'}/\mathrm{P}^s_{n\beta})(\mathrm{P}^s_{n\beta}\,\mathrm{S}^s_{n\mu}-\mathrm{P}^s_{n\mu}\,\mathrm{S}^s_{n\beta})$$

$$\theta>\theta'.$$

(3) In 31 (3) change $\cosh s(\pi-\phi+\phi')$ into

$$2\sinh s(\alpha-\phi)\sinh s\phi'\sinh s\pi/\sinh s\alpha.$$

$$\phi>\phi'.$$

35. *Space bounded by two spheres, two axial planes, and two cones*

$$\rho=0,\ \rho=c,\ \phi=0,\ \phi=\alpha\ \ \mu=\beta,\ \mu=\delta,\ \beta>\delta.$$

(1) Start from the θ form of 32. Comparison with the corresponding case of the cylindrical system enables us to write down at once

$$\mathrm{V}\sqrt{rr'}=\frac{8\pi}{c\alpha}\sum_p \sin\lambda\rho\,\sin\lambda\rho'\sum_m \sin s\phi\,\sin s\phi'.$$

$$(\mathrm{P}\beta\mathrm{S}\mu'-\mathrm{P}\mu'\mathrm{S}\beta)(\mathrm{P}\delta\mathrm{S}\mu-\mathrm{P}\mu\mathrm{S}\delta)/(\mathrm{P}\beta\,.\,\mathrm{S}\delta-\mathrm{P}\delta\mathrm{S}\beta).$$

Here the harmonics are of degree $-1/2+i\lambda$ and rank s;

$$\lambda=p\pi/c,\ s=m\pi/\alpha,\ \theta>\theta'.$$

(2) In order to deduce the r form, change the order of summation, and consider the function of n,

$$\sinh(n+\tfrac{1}{2})(c-\rho)\sinh(n+\tfrac{1}{2})\rho'/\sinh(n+\tfrac{1}{2})c$$
$$\cdot\,(\mathrm{P}\beta\mathrm{S}\mu'-\mathrm{P}\mu'\mathrm{S}\beta)(\mathrm{P}\delta\mathrm{S}\mu-\mathrm{P}\mu\mathrm{S}\delta)/(\mathrm{P}\beta\mathrm{S}\delta-\mathrm{P}\delta\,.\,\mathrm{S}\beta),$$

the harmonics being of degree n and rank s.

If $\rho>\rho'$, $\theta>\theta'$, this function, which is odd in $n+1/2$, vanishes for every infinite n; the total sum of its residues is therefore *nil*. The infinities are simple poles, namely, the (pure imaginary) zeros of $\sinh(n+1/2)c$ and the (real) zeros of $\mathrm{P}\beta\,\mathrm{S}\delta-\mathrm{P}\delta\,\mathrm{S}\beta$.

Hence from (1)

$$\mathrm{V}\sqrt{rr'}=-\frac{8\pi}{a}\sum_m \sin s\phi\,\sin s\phi'\sum_n \frac{\sinh(n+\frac{1}{2})(c-\rho)\sinh(n+\frac{1}{2})\rho'}{\sinh(n+\frac{1}{2})c}\cdot$$
$$(\mathrm{P}\beta\mathrm{S}\mu'-\mathrm{P}\mu'\mathrm{S}\beta)(\mathrm{P}\delta\mathrm{S}\mu-\mathrm{P}\mu\mathrm{S}\delta)\Big/\frac{d}{dn}(\mathrm{P}\beta\mathrm{S}\delta-\mathrm{P}\delta\mathrm{S}\beta).$$

The harmonics are of degree n, rank s;

$s=m\pi/a$; n is a zero $>-1/2$ of the function of n,

$$\mathrm{P}\beta\mathrm{S}\delta-\mathrm{P}\delta\mathrm{S}\beta\,;\ r>r'.$$

(3) Taking (1) as it stands, consider the function of s,

$$\sin s(a-\phi)\sin s\phi'/\sin sa$$
$$\cdot\,(\mathrm{P}\beta\mathrm{S}\mu'-\mathrm{P}\mu'\mathrm{S}\beta)(\mathrm{P}\delta\mathrm{S}\mu-\mathrm{P}\mu\mathrm{S}\delta)/(\mathrm{P}\beta\mathrm{S}\delta-\mathrm{P}\delta\mathrm{S}\beta),$$

the harmonics being of degree $-1/2+i\lambda$, rank s.

If $\phi>\phi'$, $\theta>\theta'$, this function, which is odd in s, vanishes for every infinite s, and has simple poles at the (pure imaginary) zeros of $\mathrm{P}\beta\mathrm{S}\delta-\mathrm{P}\delta\mathrm{S}\beta$, and at the (real) zeros of $\sin sa$.

We thus find, as before,

$$\mathrm{V}\sqrt{rr'}=-\frac{8\pi}{c}\sum_p \sin\lambda\rho\,\sin\lambda\rho'\sum_s \frac{\sinh s(a-\phi)\sinh s\phi'}{\sinh sa}\cdot$$
$$(\mathrm{P}\beta\mathrm{S}\mu'-\mathrm{P}\mu'\mathrm{S}\beta)(\mathrm{P}\delta\mathrm{S}\mu-\mathrm{P}\mu\mathrm{S}\delta)\Big/\frac{d}{ds}(\mathrm{P}\beta\mathrm{S}\delta-\mathrm{P}\delta\mathrm{S}\beta).$$

The harmonics are of degree $-1/2+i\lambda$, rank is;

$\lambda=p\pi/c$; s a positive zero of the function of s

$$\mathrm{P}\beta\mathrm{S}\delta-\mathrm{P}\delta\mathrm{S}\beta\,;\ \phi>\phi'$$

36. *Relation of solutions in the two coordinate systems.*

The reader has doubtless observed the pronounced analogy between the forms of the solutions of allied cases in the two systems of coordinates. This similarity is indeed no more than might have been expected, seeing that the cylindrical is a special limiting case of the spherical system, namely, when the centre of the spheres has gone to infinity.

In the equation $r = ae^{\rho}$, we suppose a to become infinite while $r - a$ remains finite; then

the spherical	ρ	becomes the cylindrical	z/a
,,	,,	θ	,,	,,	,,	ρ/a
,,	,,	ϕ	remains		,,	ϕ.

With the aid of these limits, and those of 15 (1), the reduction of the spherical solutions to the others can easily be made.

(Cf. Lord Kelvin and Tait, *Nat. Phil.*, Vol. II., §783. In this paragraph, which appeared in the first edition of 1867, the relation of cylindrical to spherical harmonics is clearly pointed out, and methods are indicated for arriving at formulae given explicitly, but apparently afterwards, by Neumann, Mehler, and Heine.)

Similarly from the cylindrical solutions it is possible to deduce solutions in ordinary rectangular coordinates, by sending the centre of the cylinders to infinity. The reductions are not quite so easy as in the former case, some rather troublesome limits being required, particularly the limits of $J_m\lambda\rho$ and $G_m\lambda\rho$ when m and ρ become infinite, but in a finite ratio.

For the case of $J_m\lambda\rho$ with m and λ real and positive, see Graf und Gubler, "Theorie der Bessel'schen Funktionen," Bern 1898; s. 96.

Of course the rectangular solutions can easily be found by a direct process.

37. *Semi-convergent symmetrical solutions, or Stokes' forms.*

Reference has been made in the introductory paragraph to the solution given by Stokes for a space bounded by six planes of the rectangular coordinate system.* Now for each of the functions considered in the present paper, it appears that there are *two* distinct forms, analagous to that obtained by Stokes, involving harmonics of real, and of purely imaginary rank respectively. These

* "*On the Critical Values of the Sums of Periodic Series.*" (*Collected Works*, vol. I., p. 301.)

Stokes' forms, though arithmetically much less simple than the forms already given, are of considerable analytical intertest, for this reason among others, that they are explicitly symmetrical in the accented and unaccented coordinates. They can be derived with little trouble from the expressions already found, the transformations in a large number of instances being most easily effected by means of a theorem of Cauchy's, which it may be useful to state, namely :—If the finite singularities of a uniform function $f(z)$ are all simple poles, and if, for an indefinitely increasing z, $f(z)$ tends to zero (except at the poles, if these extend to infinity, and, it may be, for certain values of the phase of z, for which $f(z)$ remains finite), then the function is equal to the sum of its polar elements, a polar element of the function being $A/(z-a)$, where a is a pole, and A the residue at that pole.

The theorem, which includes many well-known formulae of analysis, is proved at once on applying the fundamental residue theorem to the function of a, $f(a)/(z-a)$.

As examples, take the function of λ,

$$\sinh \lambda(c-z)\sinh \lambda z'/\sinh \lambda c, \quad 0<z'<z<c.$$

The conditions are satisfied, and the function is equivalent to

$$\frac{1}{c}\sum_p \sin p\pi z/c \,.\, \sin p\pi z'/c \,.\, \left(\frac{1}{\lambda - ip\pi/c} + \frac{1}{\lambda + ip\pi/c}\right)$$
$$= \frac{2}{c}\sum_p \sin p\pi z/c \,.\, \sin p\pi z'/c \,.\, \frac{\lambda}{\lambda^2 + p^2\pi^2/c^2}.$$

Again, the function of n,

$$S_n^m\mu \, P_n^m\mu' = \Pi(m+n)\Pi(m-n-1)P_n^m(-\mu)P_n^m\mu'$$

satisfies the conditions if $\theta > \theta'$.

Its poles are those of $\Pi(m+n)$ and of $\Pi(m-n-1)$,

that is $n = m+p$, $n = -m-p-1$, with p zero or a positive integer.

The residue of $\Pi(m-n-1)$ when $n = m+p$ is $(-)^{p-1}/\Pi p$, §30, (3).

Hence residue of the function when $n = m+p$ is

$$-(\overline{\Pi p + 2m}/\Pi p)P_{m+p}^m\mu \,.\, P_{m+p}^m\mu'.$$

Also since the function is even in $n+\frac{1}{2}$,

the residue at $n = -m-p-1$ is the same as this, except for sign.

Hence

$$S_n^m\mu \,.\, P_n^m\mu' = \sum_p \frac{\Pi(p+2m)}{\Pi p} P_{m+p}^m\mu \,.\, P_{m+p}^m\mu' \left(\frac{1}{n+m+p+1} - \frac{1}{n-m-p}\right).$$

Some special cases of this are interesting, *e.g.*, when $m=0$,

$$P_n(-\mu)P_n\mu' = \frac{\sin n\pi}{\pi}\sum_p\left(\frac{1}{n-p}-\frac{1}{n+p+1}\right)P_p\mu\, P_p\mu';\ \theta>\theta',$$

and putting $\theta'=0$ in this

$$P_n(-\mu) = \frac{\sin n\pi}{\pi}\sum_p\left(\frac{1}{n-p}-\frac{1}{n+p+1}\right)P_p\mu$$

so that, changing θ into $\pi-\theta$,

$$P_n(\mu) = \frac{\sin n\pi}{\pi}\sum_p\left(\frac{1}{n-p}-\frac{1}{n+p+1}\right)(-)^p P_p\mu.$$

The two last formulae evidently give the expansions of $P_n\mu$, $P_n(-\mu)$ in terms of Legendre's functions.

38. As specimens of Stokes' forms, and of the methods by which they may be found, we take the Green's function for a space bounded by two axial planes, §21.

In 21 (1) we replace $\sinh s(\alpha-\phi)\sinh s\phi'/\sinh s\alpha$ by

$$\frac{2s}{\alpha}\sum_m \sin\frac{m\pi\phi}{\alpha}\sin\frac{m\pi\phi'}{\alpha}\cdot\frac{1}{s^2+m^2\pi^2/\alpha^2},\ §37,$$

and find

$$(4)\quad V=\frac{16}{\pi^2\alpha}\int_0^\infty \cos\lambda(z-z')d\lambda\int_0^\infty e^{-s\pi}\sinh s\pi\, G_{is}i\lambda\rho\, G_{is}i\lambda\rho'.\, sds.$$

$$\sum_m \sin\frac{m\pi\phi}{\alpha}\sin\frac{m\pi\phi'}{\alpha}\cdot\frac{1}{s^2+m^2\pi^2/\alpha^2}.$$

Again in 21 (3), for $e^{-\lambda(z-z')}$ write

$$\frac{2}{\pi}\int_0^\infty \frac{\lambda\cos\kappa(z-z')d\kappa}{\kappa^2+\lambda^2},$$

and (for convenience of comparision) interchange the symbols λ, κ. Hence

$$(5)\quad V=\frac{8}{\alpha}\sum_m \sin\frac{m\pi\phi}{\alpha}\sin\frac{m\pi\phi'}{\alpha}\int_0^\infty \kappa J_{\frac{m\pi}{\alpha}}\kappa\rho\, J_{\frac{m\pi}{\alpha}}\kappa\rho'.\, d\kappa.$$

$$\int_0^\infty \frac{\cos\lambda(z-z')d\lambda}{\kappa^2+\lambda^2}.$$

In (4) we may integrate with respect to s before summing with respect to n, provided, as appears from 21 (2),

$$\frac{2}{\pi^2}\int_0^\infty \frac{s}{s^2+m^2\pi^2/a^2} e^{-s\pi} \sinh s\pi \, \mathrm{G}_{is} i\lambda\rho \, \mathrm{G}_{is} i\lambda\rho' ds$$
$$= \mathrm{G}_{\frac{m\pi}{a}} i\lambda\rho \, \mathrm{J}_{\frac{m\pi}{a}} i\lambda\rho', \quad \rho > \rho'.$$

Also in (5) we may first integrate with respect to κ, then sum with respect to m, and lastly integrate with respect to λ, provided

$$\int_0^\infty \frac{\kappa}{\kappa^2+\lambda^2} \mathrm{J}_{\frac{m\pi}{a}} \kappa\rho \, \mathrm{J}_{\frac{m\pi}{a}} \kappa\rho' d\kappa$$
$$= \mathrm{G}_{\frac{m\pi}{a}} i\lambda\rho \, \mathrm{J}_{\frac{m\pi}{a}} i\lambda\rho', \quad \rho > \rho'.$$

These two theorems are true and can easily be proved by the method of Cauchy, already exemplified so often. The second theorem is given by Sonine, *Math. Annalen* XVI.

If we were to go through the other functions in like manner, we should find suggested many interesting theorems, easy to prove, and indirectly verifying the legitimacy of altering the order of summations and integrations in the Stokes' forms.

39. *Continuation of Green's function beyond the bounded space containing the pole.*

The close analogy between Newtonian potential theory and the theory of functions of a complex variable has often been remarked. In each theory one of the most important ideas is that of the continuation or extension of a function beyond a limited region for which alone it is, in the first instance, analytically defined.

Now the transformations by which in this paper one form for a Green's function has been deduced from another have all this interesting characteristic that they carry on the function to a region in some way more extensive than that for which the original form is valid. This extension is valuable for various reasons, particularly as an aid to the determination of the singularities of the complete function.

Thus in §19, the form (1) defines a potential function for the space $z' < z < 2c - z'$; the form (2) extends the function to the space

$\rho > \rho'$, for all values of z. Moreover, this second form, with the cognate form for $\rho < \rho'$, shows that the function is odd and periodic in z, with period $2c$. Hence knowing the one singularity of the function between the limits $o < z < c$, we can infer the position and nature of the singularities of the complete function, and obtain the results of the method of images. When the boundary is cylindrical or conical, the singularities are of a more complex character, but they may be investigated from the materials given here, as I hope to show later.

40. *Sommerfeld's Function.*

Not the least interesting cases are those in which the boundary consists in part of two axial planes, the special feature being that the extended function, as a function of ϕ, has the period 2α instead of 2π, so that it does not in general return to its original value when the variable point makes a complete circuit about the axis of z. In order to obtain a quasi-geometrical representation of such a function, Professor Sommerfeld introduces the conception of a Riemann's space winding about the axis of z, a space in which we may consider functions not necessarily periodic in ϕ, whose value is supposed to range from $-\infty$ to $+\infty$.

[*Proceedings of the London Mathematical Society*, Vol. 28, "Über verzweigte Potentiale im Raum." See also in Vol. 30 of the same *Proceedings*, "Some multiform solutions of the Partial Differential Equations of Physical Mathematics," by Dr H. S. Carslaw, and compare Lord Kelvin and Tait, *Nat. Phil.*, Vol. I., App. B (c)].

In such a space the function 1/ distance or T is not the simplest conceivable potential, for it has a singularity of the first order, not for $\phi = \phi'$ only but for $\phi = \phi' + 2n\pi$ where n is any integer.

The fundamental potential, or Sommerfeld's function, as we may call it, will have but one singularity in the whole space, namely, at some point whose $\phi = \phi'$. (The axis of z is supposed excluded from the space by a very fine winding cylinder.)

Expressions for this function in terms of cylindrical and spherical harmonics may be found without difficulty. The easiest way, from our present standpoint, is to take one of the solutions for a space bounded by two axial planes, say $\phi = \pm \alpha/2$, and find its limiting

form as a increases indefinitely. Thus in 21 (1), when the plane from which ϕ is measured is turned through an angle $a/2$, so that the factor involving a becomes

$$\sinh s\left(\frac{a}{2}-\phi\right)\sinh s\left(\phi'+\frac{a}{2}\right)\Big/\sinh sa,$$

we may, as it is easy to show, find the limit of V by writing its limit instead of this factor; this limit is $\frac{1}{2}e^{-s(\phi-\phi')}$. Hence

$$(1)\quad \mathrm{V}=\frac{4}{\pi^2}\int_0^\infty \cos\lambda(z-z')d\lambda\int_0^\infty e^{-s\pi}\sinh s\pi\,.\,e^{-s(\phi-\phi')}\,\mathrm{G}_{is}i\lambda\rho\,\mathrm{G}_{is}i\lambda\rho' ds. \qquad \phi>\phi'.$$

(2) The second integral here is

$$-\int e^{im\pi}\sin m\pi\, e^{im(\phi-\phi')}\mathrm{G}_m i\lambda\rho\,\mathrm{G}_m i\lambda\rho' dm \;\Big|\; \mathrm{ON}$$

$$=\;\frac{\pi}{2}\int e^{im(\phi-\phi')}(i^m\mathrm{G}_m i\lambda\rho)(i^{-m}\mathrm{J}_m i\lambda\rho')dm \;\Big|\; \mathrm{ON}$$

$$-\frac{\pi}{2}\int e^{im(\phi-\phi')}(i^m\mathrm{G}_m i\lambda\rho)(i^m\,\mathrm{J}_{-m}i\lambda\rho')dm \;\Big|\; \mathrm{ON}.$$

If $\rho>\rho'$, $\phi>\phi'$, the path of integration in the former integral may be changed to OE, in the latter to OW. Expressing these as real integrals, and collecting, we find

$$\mathrm{V}=\frac{4}{\pi}\int_0^\infty \cos\lambda(z-z')d\lambda\int_0^\infty \cos m(\phi-\phi')\mathrm{G}_m i\lambda\rho\,\mathrm{J}_m i\lambda\rho' dm. \qquad \rho>\rho'$$

(3) The z form is deduced from this as in 21 (3),

$$\mathrm{V}=2\int_0^\infty \cos m(\phi-\phi')dm\int_0^\infty e^{-\lambda(z-z')}\mathrm{J}_m\lambda\rho\,\mathrm{J}_m\lambda\rho' d\lambda. \qquad z>z'.$$

The Stokes' forms may now be written down as in 38 (4), and the corresponding expressions in spherical harmonics may easily be obtained in the same way.

41. If we denote the above function by $\mathrm{V}(\phi)$, then a function having singularities of the first order at ϕ, $\phi\pm 2a$, $\phi\pm 4a$, etc, is

$$\mathrm{W}=\mathrm{V}(\phi)+\mathrm{V}(\phi+2a)+\mathrm{V}(\phi+4a)+\ldots\ldots$$
$$+\mathrm{V}(\phi-2a)+\mathrm{V}(\phi-4a)+\ldots\ldots$$

The form 40 (1) is convenient for calculating this series. We obtain

$$(1)\quad W=\frac{4}{\pi^2}\int_0^\infty \cos\lambda(z-z')d\lambda \int_0^\infty e^{-s\pi}\sinh s\pi\frac{\cosh s(\alpha+\phi'-\phi)}{\sinh s\alpha}G_{is}i\lambda\rho G_{is}i\lambda\rho' ds$$

$$o<\phi-\phi'<2\alpha.$$

(2) From this we deduce as usual

$$W=\frac{4}{\alpha}\int_0^\infty \cos\lambda(z-z')d\lambda\ \Sigma'_m \cos\frac{m\pi}{\alpha}(\phi-\phi')G_{\frac{m\pi}{\alpha}}i\lambda\rho\, J_{\frac{m\pi}{\alpha}}i\lambda\rho',$$

$$\rho>\rho'.$$

$$(3)\quad W=\frac{2\pi}{\alpha}\int_0^\infty e^{-\lambda(z-z')}d\lambda\ \Sigma'_m \cos\frac{m\pi}{\alpha}(\phi-\phi')J_{\frac{m\pi}{\alpha}}\lambda\rho\,.\,J_{\frac{m\pi}{\alpha}}\lambda\rho'.$$

$$z>z'.$$

The functions T and W coincide when $\alpha=\pi$.

42. The relation of the solution of Green's problem to the general problem of determining a potential function taking an arbitrary value at a given boundary is well known. When this paper was commenced, it was intended to consider in some detail, and to illustrate from the foregoing solutions, certain questions arising in this connection, particularly with reference to modes of expansion of an arbitrary function. Since, however, the paper is already sufficiently long, these questions must be left over for the present, but I propose to deal with them in a future paper, which I hope to have the privilege of reading to the Society at one of next session's meetings.

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On the Discriminant of the General Homogeneous Quadric.

By Chas. Tweedie, M.A., B.Sc., F.R.S.E.

A note on change of Coordinate Axes.

By Prof. Steggall.

Sixth Meeting, May 11th, 1900.

R. F. MUIRHEAD, Esq., M.A., B.Sc., President, in the Chair.

Note on a Theorem in Continued Fractions.

By Prof. STEGGALL.

Note on the Fundamental Inequality Theorems connected with e^x and x^m.

By Prof. GEORGE A. GIBSON.

The subject of this note is that dealt with in Mr Tweedie's paper in the *Proceedings*, vol. XVII., 33–37, and my only reason for bringing it before the Society is to call attention to a slightly different method of presenting the same order of ideas. The method is that adopted by Peano, *Lezioni di Analisi Infinitesimale*, vol. I., §23, but as the book is not readily accessible to teachers, there may be some interest in having the method reproduced in our *Proceedings*. I add one or two remarks.

Peano starts, as Mr Tweedie does, from the generalised arithmetico-geometrical mean, namely, that if a, b, m, n be any positive quantities and a not equal to b,

$$a^m b^n < \left(\frac{ma+nb}{m+n}\right)^{m+n}.$$

His procedure is as follows:—Let $a = 1 + 1/m$, $b = 1$ and we get

$$\left(1+\frac{1}{m}\right)^m < \left(1+\frac{1}{m+n}\right)^{m+n} \quad \cdots \quad (1)$$

Next let $a=1$, $b=1-1/n$ ($n>1$, so that b may be positive) and we get

$$\left(1-\frac{1}{n}\right)^n < \left(1-\frac{1}{m+n}\right)^{m+n},$$

that is,
$$\left(1-\frac{1}{n}\right)^{-n} > \left(1-\frac{1}{m+n}\right)^{-(m+n)} \quad \cdots \quad (2)$$

Then, let $a=1+1/m$, $b=1-1/n$ and we get

$$\left(1+\frac{1}{m}\right)^m\left(1-\frac{1}{n}\right)^n < 1,$$

that is,
$$\left(1+\frac{1}{m}\right)^m < \left(1-\frac{1}{n}\right)^{-n} \quad \cdots \quad (3)$$

Equation (1) shows that the expression $(1+1/z)^z$ increases as z increases, provided z is positive, while equation (2) shows that when z is negative, the expression decreases as z increases in absolute value; equation (3) proves that for any positive value of z, the expression is less than for any negative value of z (z being numerically greater than unity).

Finally, to show that when z increases indefinitely in absolute value the expression $(1+1/z)^z$ has the same limit whether z be positive or negative, put $m+1$ for n in equation (3); then

$$\left(1-\frac{1}{n}\right)^{-n} = \left(\frac{m}{m+1}\right)^{-(m+1)} = \left(1+\frac{1}{m}\right)^m\left(1+\frac{1}{m}\right).$$

Hence
$$\operatorname*{Limit}_{n=\infty}\left(1-\frac{1}{n}\right)^{-n} = \operatorname*{Limit}_{m=\infty}\left(1+\frac{1}{m}\right)^m \times \operatorname*{Limit}_{m=\infty}\left(1+\frac{1}{m}\right)$$

$$= \operatorname*{Limit}_{m=\infty}\left(1+\frac{1}{m}\right)^m;$$

so that
$$\left(1+\frac{1}{m}\right)^m < e < \left(1+\frac{1}{m}\right)^{m+1}.$$

It is obvious that Peano's method may at once be extended to the limits for e^x whether x be positive or negative, it being remembered that $1+x/m$, $1-x/n$ must always be positive. Thus we have

$$\left(1+\frac{x}{m}\right)^m < e^x < \left(1-\frac{x}{n}\right)^{-n} \quad - \quad - \quad - \quad - \quad (4)^*$$

or
$$\left(1+\frac{x}{m}\right)^m < e^x < \left(1+\frac{x}{m}\right)^m \left(1+\frac{x}{m}\right)^x,$$

by putting $m+x$ for n.

The standard inequalities for $e^{\pm x}$, $\log(1 \pm x)$ follow at once.

Again if x be numerically less than unity, equation (4) gives, by putting $m = 1 = n$,

$$x < e^x - 1 < \frac{x}{1-x},$$

and $\therefore$
$$\underset{x=0}{\text{Limit}} \quad \frac{e^x-1}{x} = 1.$$

With reference to the inequalities

$$1 + mx^{m-1}(x-1) \lesseqgtr x^m \lesseqgtr 1 + m(x-1),$$

it is perhaps worth being more carefully emphasized that when any one of the four has been proved generally the other three follow by simple substitutions. One of the inequalities is given at once by the inequality

$$a^m b^n < \left(\frac{ma+nb}{m+n}\right)^{m+n}.$$

Put $b = 1$, $n = 1-m$, so that m is a positive proper fraction, and we get

$$a^m < 1 + m(a-1) \quad - \quad - \quad - \quad - \quad (\alpha)$$

Write $1/a$ for a and we get, after multiplying by a^m

$$a^m > 1 + ma^{m-1}(a-1) \quad - \quad - \quad - \quad (\beta)$$

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* The reason for writing e^x as the function of x which lies between $(1+x/m)^m$ and $(1-x/n)^{-n}$ is that

$$\left(1+\frac{x}{m}\right)^m = \left\{\left(1+\frac{1}{\mathrm{M}}\right)^{\mathrm{M}}\right\}^x,$$

where $\mathrm{M} = mx$. The inequalities hold whether x be positive or negative, but m, n must be positive.

Again in (α) for m write $1/m$ so that m is now greater than unity, and we get

$$a^{\frac{1}{m}} < 1 + \frac{a-1}{m} .$$

For a write now a^m and we get

$$a^m > 1 + m(a-1) \quad - \quad - \quad - \quad - \quad (\gamma)$$

For a write $1/a$ and we get

$$a^m < 1 + ma^{m-1}(a-1) \quad - \quad - \quad - \quad (\delta)$$

So far the index is necessarily positive; but if m be any negative number, $1/(1-m)$ will be a positive proper fraction.
In (α) write $1/(1-m)$ for m and a^{1-m} for a and we get

$$a^m < 1 + ma^{m-1}(a-1),$$

which is (δ). If we now write $1/a$ for a we get (γ).

In these inequalities the important variable is the index. If we write x in place of m and compare the graphs of a^x and $1 + x(a-1)$ we see (i) that the graphs intersect at $x=0$ and $x=1$; (ii) that by (α) the ordinate of a^x is less than the corresponding ordinate of the line $1 + x(a-1)$, so long as x lies between 0 and 1; (iii) that by (γ) the ordinate of a^x is greater than the corresponding ordinate of the line $1 + x(a-1)$ for x lying outside the range from 0 to 1. The geometrical representation may aid the memory in retaining the inequalities, as the graph of the exponential curve is about as familiar as that of the straight line.

Note sur un Problème de Géométrie,

par Ed. Collignon,

Inspecteur Général des ponts et chaussées, à Paris.

Etant donné un triangle OAB, on demande de mener par le sommet O une droite OM telle qu' en abaissant les perpendiculaires Aa, Bb sur cette droite, les surfaces des triangles OAa, OBb soient entre elles dans un rapport r donné, c'est-à-dire qu'on ait la relation

$$r = \frac{\text{Surf. OA}a}{\text{Surf. OB}b}.$$

Solution analytique.

Figure 8.

Prenons pour axe des abscisses OX une parallèle menée par le sommet O à la base AB ; et pour axe OY la hauteur OH perpendiculaire à cette base indéfiniment prolongée.

Soit $OH = h$ la distance des deux parallèles AB, OX ; h sera l'ordonnée commune aux points A et B.
Soient $HA = p$ $HB = q$ leurs abscisses respectives.

Soit $y = \lambda x$ l'équation de la droite cherchée OM.
L'inconnue de la question sera le coefficient λ.

Les droites Aa, Bb auront pour équations

$$\text{A}a \qquad y - h = -\frac{1}{\lambda}(x - p),$$

$$\text{B}b \qquad y - h = -\frac{1}{\lambda}(x - q).$$

Les segments Aa, Bb, Oa, Ob seront donnés en valeur absolue par les relations

$$\text{A}a = \frac{h - \lambda p}{\sqrt{1 + \lambda^2}}, \qquad \text{B}b = \frac{h - \lambda q}{\sqrt{1 + \lambda^2}},$$

$$\text{O}a = \frac{h\lambda + p}{\sqrt{1 + \lambda^2}}, \qquad \text{O}b = \frac{h\lambda + q}{\sqrt{1 + \lambda^2}},$$

dans lesquelles les seconds membres doivent être pris positivement ; et l'on aura pour déterminer λ l'équation du second degré

$$(1) \qquad r = \frac{Aa \times Oa}{Bb \times Ob} = \frac{(h - \lambda p)(p + \lambda h)}{(h - \lambda q)(q + \lambda h)},$$

où l'on peut convenir de prendre en valeur absolue chacun des facteurs du second membre. Mais il paraît préférable d' attribuer au rapport r le signe que lui assigne l'équation (1) elle-même, en laissant à chaque facteur son signe particulier. Cela revient à admettre pour le rapport donné r des valeurs positives et négatives. On reconnaît sur le champ que r change de signe chaque fois que λ traverse l'une des valeurs $\frac{h}{p}, \frac{h}{q}, -\frac{p}{h}, -\frac{q}{h}$.

Si l'on appelle λ' et λ'' les deux racines de l'équation (1) résolue par rapport à λ, on reconnaît aisément que le produit $\lambda'\lambda'' = -1$, car l'équation (1) ne change pas quand on y change λ en $-\frac{1}{\lambda}$.

Si une droite OM satisfait à la condition proposée, la droite ON perpendiculaire à OM y satisfait également.

Les triangles OAa', OBb' sont en effet respectivement égaux aux triangles OAa, OBb, et ont entre eux le même rapport.

Faisons par exemple $\lambda = 0$, puis $\lambda = \infty$, ce qui définit les deux axes OX OY ; ces hypothèses conduisent toutes deux à $r = \frac{p}{q}$. Faisons ensuite $\lambda = \pm 1$, ce qui correspond aux bissectrices de l'angle des axes ; on aura $r = \frac{p^2 - h^2}{q^2 - h^2}$.

Les trois points M, O, N, sommets du triangle rectangle MON, sont situés sur une circonférence dont le centre est au point P milieu de l'hypoténuse MN. Cherchons les abscisses des points M et N.

Appelons x d'une manière générale l'abscisse d'un point situé sur la droite AB, correspondante à la valeur λ du coefficient d'inclinaison de la droite qui joint ce point à l'origine. Nous aurons

$$h = \lambda x,$$

relation qui permet de remplacer λ dans l'équation (1) par sa valeur $\frac{h}{x}$.

Il vient pour déterminer les x des points M et N l'équation

$$r = \frac{(x-p)(px+h^2)}{(x-q)(qx+h^2)},$$

ou bien, en ordonnant par rapport à x,

$$x^2 - \frac{(p^2-h^2) - r(q^2-h^2)}{p-qr}x - h^2 = 0. \tag{2}$$

La somme algébrique des abscisses des points M et N est égale à $\frac{p^2-h^2-r(q^2-h^2)}{p-qr}$ coefficient de x dans le premier membre de l'équation. L'abscisse $u = \mathrm{HP}$ du point P en est la moitié, et l'on a

$$u = \frac{p^2-h^2-r(q^2-h^2)}{2(p-qr)}. \tag{3}$$

Résolvons cette équation par rapport à r. Il vient

$$r = \frac{p}{q}\left(\frac{u - \frac{p^2-h^2}{2p}}{u - \frac{q^2-h^2}{2q}}\right) = \frac{p}{q}\left\{\frac{u - \frac{1}{2}\left(p - \frac{h^2}{p}\right)}{u - \frac{1}{2}\left(q - \frac{h^2}{q}\right)}\right\} \tag{4}$$

Sous cette forme l'équation conduit à une construction géométrique simple du point P. Remarquons en effet :

Que p est l'abscisse du point A et que $-\frac{h^2}{p}$ est l'abscisse du point A′ que l'on obtient, sur la droite AB prolongée, en la coupant par une droite OA′ élevée au point O perpendiculairement à OA ; par conséquent $\frac{1}{2}\left(p - \frac{h^2}{p}\right)$ définit sur la droite AB un point α, milieu de AA′, et centre de la circonférence qui passe par les trois points A, O, A′ ;

Que, de même, $\frac{1}{2}\left(q - \frac{h^2}{q}\right)$ est l'abscisse d'un point β, milieu du segment BB′ que l'on obtient en menant au point O une perpendiculaire à OB jusqu' à la rencontre de AB prolongée.

Les différences $u - \frac{1}{2}\left(p - \frac{h^2}{p}\right)$, $u - \frac{1}{2}\left(q - \frac{h^2}{q}\right)$ représentent les distances $\mathrm{P}\alpha$, $\mathrm{P}\beta$ du point P à ces deux points α et β, et l'équation (4) revient à l'égalité

$$(5) \qquad \frac{P\alpha}{P\beta} = r \times \frac{q}{p} = r',$$

en appelant r' le rapport connu $r \times \frac{q}{p}$.

On trouvera donc la position du point P en cherchant sur la droite $\alpha\beta$ le point P tel que le rapport de ses distances aux points connus α et β soit égal à un nombre r' déduit du rapport r par l'opération $r' = \frac{qr}{p}$.

Nous avons vu qu'on devait attribuer à r le signe +, puis le signe −, pour prévoir tous les cas possibles. Le signe attribué à r entraine un signe particulier pour r', de sorte que r' doit aussi recevoir successivement le signe + et le signe −. A chacun correspond une position du point P sur $\alpha\beta$.

FIGURE 9.

L'équation (5) dans laquelle r' reçoit le double signe définit deux positions P_1 et P_2 du point cherché, et ces deux positions sont conjuguées harmoniques par rapport aux extrémités du segment $\alpha\beta$ qu'elles partagent.

Connaissant le point P on achèvera la solution en décrivant du point P comme centre avec PO pour rayon une circonférence qui coupera AB en M et N. Les droites cherchées OM ON sont parallèles aux bissectrices des deux angles que la droite OP fait avec la base AB du triangle. Comme il y a deux positions pour le point P et qu' à chacune correspondent deux droites OM, ON, on a en tout quatre solutions.

Lorsque $r = 0$, r' est nul aussi et le point P coïncide avec α ; les droites cherchées coïncident avec OA, OA'.

Lorsque r est infini, r' l'est également et le point P coïncide avec le point β ; les droites correspondantes sont OB, OB'.

La solution générale dérive, en définitive, d'une combinaison des deux solutions particulières relatives à $r = 0$ et à r infini.

On remarquera que les points A, A', B, B', M et N forment une *involution*.

Faisons, à titre d'exemple, $r = 1$. On en déduit $r' = \frac{q}{p}$, et l' équation (5) devient

$$\frac{P\alpha}{P\beta} = \frac{HB}{HA},$$

d'où l'on déduit cette propriété que les droites $\alpha\beta P$, BAH sont divisées proportionellement aux points α, β, P et B, A, H.

Il est plus simple de chercher la valeur de u dans l'équation (3). On a pour $r = 1$

$$u = \frac{p^2 - q^2}{2(p-q)} = \frac{p+q}{2},$$

de sorte que le point P est le milieu de la base AB.

Les droites OM, ON qui assurent l'équivalence des triangles OAa OBb sont donc parallèles aux bissectrices des angles que la médiane OI forme avec la base du triangle.

Si l'on fait $r = -1$ on obtient une seconde solution qui assure la même équivalence. L'équation (3) donne alors

$$u = \tfrac{1}{2} \cdot \frac{p^2 + q^2 - 2h^2}{p+q} = \tfrac{1}{2}(p+q) - \frac{h^2 + pq}{p+q}.$$

FIGURE 10.

On n'a donc qu' à porter dans le sens convenable une quantité IP_2 égale à $-\frac{h^2+pq}{p+q}$ à partir du milieu I de AB, pour avoir une autre solution ; les droites correspondantes sont les parallèles aux bissectrices des angles que forme OP_2 avec la droite AB prolongée.

Faisons l'application de cette théorie à quelques cas particuliers.

TRIANGLE ISOSCÈLE.

FIGURE 11.

Nous aurons $q = -p$; il viendra
pour $r = 1$ $u = 0$,
pour $r = -1$ u infini.

A ces deux déterminations de u correspondent quatre droites :

Les deux parallèles aux bissectrices des angles que la médiane OH forme avec la base AB ;

Les deux droites rectangulaires OH et OX parallèles respectivement à la médiane et à la base.

Ces quatre droites font autour du point O huit angles consécutifs égaux à la moitié d'un angle droit.

Triangle rectangle en O.

Figure 12.

Les abscisses p et q seront de signes contraires, et l'on aura $pq = -h^2$.

Si l'on fait la construction, le point A′ coïncidera avec le point B, et le point B′ avec le point A. Les deux points α et β seront confondus au milieu I de la base AB. Le point P défini par le rapport de ses distances à α et à β est donc indéterminé, et en quelque point qu'on le place, le rapport $\frac{P\alpha}{P\beta} = 1$. Donc $r' = 1$ et $r = \frac{p}{q}$.

La figure justifie ce résultat. Quelle que soit la droite OM sur laquelle on projette les points A et B, les triangles OAa, OBb sont semblables, et leurs surfaces sont entre elles comme les carrés $\overline{OA}^2$, $\overline{OB}^2$ des côtés homologues.

Or, dans le triangle rectangle, $\frac{\overline{OA}^2}{\overline{OB}^2} = \frac{HA}{HB} = \frac{p}{q}$ en valeur absolue.

Si l'on place le point P en I, les droites cherchées coïncident avec OA, OB et le rapport des surfaces des triangles, qui s'annulent à la fois, prend la forme indéterminée $\frac{0}{0}$.

Nous terminerons ce paragraphe par une remarque générale.

Figure 13.

Si par les points a et b, projections de A et de B sur OM on mène des droites aω, bω′ respectivement parallèles aux côtés OA, OB du triangle, ces parallèles coupent la médiane OI en des points ω et ω′, et le rapport $\frac{O\omega}{O\omega'}$ *est égal au rapport r des surfaces des triangles OAa, OBb.*

En effet le triangle OAa est équivalent au triangle OAω, le triangle OBb est équivalent à OBω′; et ces deux triangles OAω, OBω′, ayant pour hauteurs les distances égales des points A et B à la médiane OI, sont entre eux comme leurs bases

$O\omega$, $O\omega'$. Il en résulte que, si $r = 1$, ou bien les points ω et ω' coïncident, ou bien ils sont à égale distance du sommet O.

Remarques Diverses.

Prenons le rapport r égal à l'unité positive.

Figure 14.

Les solutions correspondantes sont fournies par les droites OM, ON parallèles aux bissectrices IH, IC des angles formés par la médiane OI avec la base AB.

Les triangles NIH, OIH, rectangles en H, ont un côté commun IH et des angles égaux en I. Donc IN = IO. On prouverait de même qu'on a IM = IO ; de sorte que I est à la fois le milieu de la base AB et de la droite MN. Il en résulte que les points A et B sont situés sur une hyperbole équilatère, qui a pour asymptotes les droites OM, ON, et dans laquelle OI est la direction conjuguée aux cordes parallèles à AB. Si l'on appelle x et y les coordonnées d'un point de la courbe rapportée aux axes rectangulaires OM, ON, l'équation de l'hyperbole sera

$$xy = \text{constante}.$$

Or, pour le point A $x = Oa$, $y \equiv aA$;
pour le point B $x = Ob$, $y = bB$.

L'équation de la courbe prouve donc l'égalité

$$Oa \times aA = Ob \times bB,$$

c'est-à-dire l'équivalence des triangles OaA, ObB.

Il est aisé de démontrer cette égalité par la simple géométrie.

Le point I étant le milieu de AB, C est le milieu de ab, et IC est la demi-somme des deux ordonnées Aa, Bb, de même que OC est la demi-somme des deux abscisses Oa, Ob. On a donc

$$Oa = OC - Cb, \qquad Ob = OC + Cb,$$
$$aA = IC + IK, \qquad bB = IC - IK ;$$

et par conséquent

$$\begin{aligned} Oa \times aA - Ob \times bB &= OC \times IC - Cb \times IK + OC \times IK - Cb \times IC \\ &\quad - OC \times IC + Cb \times IK + OC \times IK - Cb \times IC \\ &= 2(OC \times IK - Cb \times IC). \end{aligned}$$

Le second membre de cette égalité est identiquement nul.

En effet, appelons θ l'angle AIH, moitié de l'angle AIO. Nous aurons

$$OC = OI\cos\theta, \qquad CI = OI\sin\theta,$$
$$IK = IB\sin\theta, \qquad Cb = IB\cos\theta.$$

Donc

$$OC \times IK - Cb \times IC = (OI \times IB - IB \times OI)\sin\theta\cos\theta = 0,$$

et le théorème est démontré.

On peut modifier cette démonstration de manière à éviter l'emploi des lignes trigonométriques. Nous devons la nouvelle démonstration que nous allons donner à l'obligeance de notre collègue et ami, M. John S. Mackay, professeur à l'Académie d' Edimbourg.

On observera d'abord que les deux droites Aa, Bb' se coupent en un point R de la médiane OI, et que Aa', Bb se coupent de même en un point S situé sur le prolongement de la même médiane ; de sorte que la figure RBSA est un rectangle, dans lequel les demi-diagonales IA, IB, IR, IS sont égales. La figure $ObSa'$ est de même un rectangle, dans lequel les droites Aa, Bb' parallèles respectivement aux deux côtés se coupent en un point de la diagonale. On a donc l'égalité

$$b'R \times RA = Ra \times ab,$$

qui exprime l'équivalence des deux rectangles partiels $b'RAa'$ et $bBRa$. Ajoutant à chacun le rectangle $OaRb'$ il y a encore équivalence des deux sommes, c'est-à-dire l'égalité

$$Oa \times aA = Ob \times bB.$$

Solution trigonométrique du cas général.

Figure 15.

Soit OAB le triangle donné, α l'angle connu AOB, μ l'angle inconnu AOM.

Nous aurons $MOB = \alpha - \mu$. Il viendra, si l'on appelle a le côté OA, b le côté OB,

$$OA' = a\cos\mu, \qquad OB' = b\cos(\alpha - \mu),$$
$$A'A = a\sin\mu, \qquad B'B = b\sin(\alpha - \mu),$$

et il viendra pour le rapport des deux aires

$$(1) \qquad r = \frac{a^2\sin\mu\cos\mu}{b^2\sin(\alpha-\mu)\cos(\alpha-\mu)} = \frac{a^2\sin 2\mu}{b^2\sin 2(\alpha-\mu)},$$

La recherche de l'angle μ revient au partage de l'angle donné 2α

But if one or more of the lines OP, PQ, OQ be not positive, equation (2) is no longer true as it stands, for in defining the cosine of an angle it is explicitly stated that the radius-vector forming one of the bounding arms of the angle (that is, in the case of the three angles in equation (2) the lines OQ, OP, PQ) is to be considered positive; manifestly, since equation (2) holds when OQ, OP, PQ are expressed by means of positive numbers, it will no longer be true if one of them be expressed by a negative number—unless some further change be made. It is in the treatment of this phase of the question that I think our text-books should be a little more explicit.

The additional explanation required is, of course, easy to give. When a segment has a sign, then the positive direction of the line on which the segment is taken must have been previously fixed; all that is required to adapt equation (2) to the case of negative segments is to take as the angle which the segment makes with the axis the angle which the positive direction of the line makes with the axis. Thus if OP is a negative segment, we have

$$\begin{aligned} OP' &= (-OP) \times \cos(OX, OP), \text{ since } (-OP) \text{ is positive} \\ &= (-OP) \times [-\cos(OX, PO)], \text{ since } (OX, PO) = (OX, OP) + 180^\circ \\ &= OP\cos(OX, PO); \end{aligned}$$

or, in words, the projection of a directed segment on a directed line is measured by the product of the number which measures the segment and the cosine of the angle which the positive direction of the segment makes with the positive direction of the line.

The general theorem corresponding to equation (2) may then be stated thus:—If OAB...KL be a broken line joining the points O, L and α, β,...λ, ω the angles which the positive directions of the lines OA, AB,...KL, OL make with the positive direction of an axis X'OX, and if a, b,...l, x measure the segments OA, AB...KL, OL in magnitude and in sign, then

$$x\cos\omega = a\cos\alpha + b\cos\beta + \ldots + l\cos\lambda.$$

In applying the theorem in trigonometry or in coordinate geometry the chief point requiring attention is that of the positive direction of the segments; when that has been settled there is no difference in carrying out the proof for different figures.

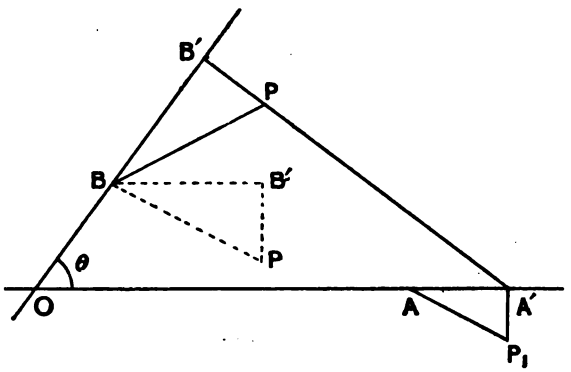
The method of projections is so fundamental in trigonometry and coordinate geometry that no pains should be spared to make its foundations clear.

Problème de la section de raison

par C. A. LAISANT, Docteur ès Sciences

La question est la suivante :

On donne deux droites OA, OB ; un point (A, B) sur chacune d'elles ; un point P dans le plan. On demande de mener par P une sécante A'B' telle que le rapport $\frac{AA'}{BB'}$ *soit égal à un rapport donné.*



Le problème étant supposé résolu, considérons les deux triangles directement semblables BB'P, AA'P_1 ; les angles B'BP, A'AP_1 sont égaux ; de plus $\frac{AP_1}{BP} = \frac{AA'}{BB'}$, qui est donné.

Donc nous pouvons construire P_1. Ceci fait, remarquons que les deux triangles sont orientés de telle sorte qu'en faisant tourner BB'P de l'angle $BOA = \theta$, ses côtés deviendraient parallèles à ceux de AA'P_1 ; l'angle de B'P avec A'P est donc θ, et par suite $PA'P_1 = \pi - \theta$. En décrivant sur PP_1 un segment capable de $\pi - \theta$ on aura donc le point A', ce qui résout le problème.

Pour abréger, je laisse de côté la discussion, qui est cependant intéressante, mais ne présente pas de difficultés fondamentales.

Note referring to the paper on page 5 of this Volume.

By R. F. MUIRHEAD, M.A., B.Sc.

When I read to the Society a paper on "*The Dissection of any two Triangles into mutually similar pairs of Triangles*," I was not aware that such a problem had ever before been considered; but it has recently been pointed out to me by Dr Mackay that in a work by W. Wallace, entitled "Geometrical Theorems and Analytical Formulae...," Edinburgh, 1839, a solution of the problem is given which is identical with the first solution in my paper. This solution appears on page 3 of Wallace's work as Proposition I., and the properties of the figure obtained are used in resolving the problem given as Proposition IV., on page 11, which is thus enunciated:— "*Three stations A, B, C being given, and the angles which the lines joining them subtend at D, a fourth station on their plane, to determine by analytical formulae the position of that fourth station.*" Other geometrical developments connected with the construction are also given.

A slight extension of Euler's Theorem on Homogeneous Functions.

By W. E. Philip, M.A.

Euler's Theorem may be looked upon as the result of a certain operator acting on a special kind of function. This function may depend on any number of variables, but for convenience it is usual to consider three, viz., x, y, z. A function is homogeneous in x, y, z and of the nth degree if it can be put into the form

$$x^n f\left(\frac{y}{x}, \frac{z}{x}\right).$$

If we adopt for conciseness the following notation, viz.,

$$\Delta_1 = x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dz} = xd_1 + yd_2 + zd_3 \quad \text{say}$$

$$\Delta_2 = (xd_1 + yd_2 + zd_3)^2$$

and generally $\Delta_p = (xd_1 + yd_2 + zd_3)^p$ where the multinomial function is to be expanded and then interpreted as an operator, we may state Euler's theorem thus

$$\Delta_p u = n(n-1)(n-2)\ldots(n-p+1)u,$$

where u is a homogeneous function of x, y, z of degree n.

In this note we wish to express the result of the same operator acting on any function of u, say $U \equiv F(u)$.

Now

$$\Delta_1 U = (xd_1 + yd_2 + zd_3)F(u) = F'(u)\left\{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right\} = nuF'(u)$$

$$= nu\frac{d}{du}F(u) = F_1(u), \text{ say.}$$

$$\Delta_1^2 U = \Delta_1 F_1(u) = nu\frac{d}{du}F_1(u) = \left(nu\frac{d}{du}\right)^2 F(u),$$

and generally

$$\Delta_1^m U = \left(nu\frac{d}{du}\right)^m F(u).$$

If we make the substitution $u = e^{n\theta}$, this gives

$$\Delta_1^m F(u) = \left(\frac{d}{d\theta}\right)^m F(e^{n\theta}) = \delta^m F(e^{n\theta}), \quad \text{where } \delta \equiv \frac{d}{d\theta}.$$

Hence it follows that any rational integral function of Δ_1 may be expressed in terms of δ.

Thus $$\phi(\Delta_1)F(u) = \phi(\delta)F(e^{n\theta}).$$

Now let $\Delta_p U$ be denoted by U_p.

We have $$\Delta_1 U_p = U_{p+1} + pU_p.$$

For we have first to apply the operator $xd_1 + yd_2 + zd_3$ to the various powers of d_1, d_2, d_3 in the expression Δ_p, and this produces U_{p+1}; then to the various powers of x, y, z, and this, by Euler's theorem, gives pU_p.

Thus $$U_{p+1} = (\Delta_1 - p)U_p.$$

Again $$U_p = (\Delta_1 - \overline{p-1})U_{p-1}$$

and $$U_1 = \Delta_1 U$$

$\therefore$ $$U_{p+1} = (\Delta_1 - p)(\Delta_1 - \overline{p-1})\ldots\Delta_1 U$$

$$= \phi(\Delta_1)F(u) = \phi(\delta)F(e^{n\theta})$$

$$= \delta(\delta - 1)\ldots(\delta - p)F(e^{n\theta}).$$

Now put $\lambda = e^{\theta}$ and we get by a well-known theorem

$$U_{p+1} = \lambda^{p+1}\frac{d^{p+1}}{d\lambda^{p+1}}F(\lambda^n).$$

This is the result desired.

Suppose, for example, $F(u) = u$

then $$U_p = \lambda^p \frac{d^p}{d\lambda^p}(\lambda^n) = n(n-1)\ldots(n-p+1)\lambda^n$$

$$= n(n-1)(n-p+1)u, \quad \text{Euler's result.}$$

Again take $$F(u) \equiv \log u = \log\lambda^n = n\log\lambda$$

$$U_p = (-1)^{p+1}(p-1)!$$

The case where u is homogeneous and of the first degree gives the result

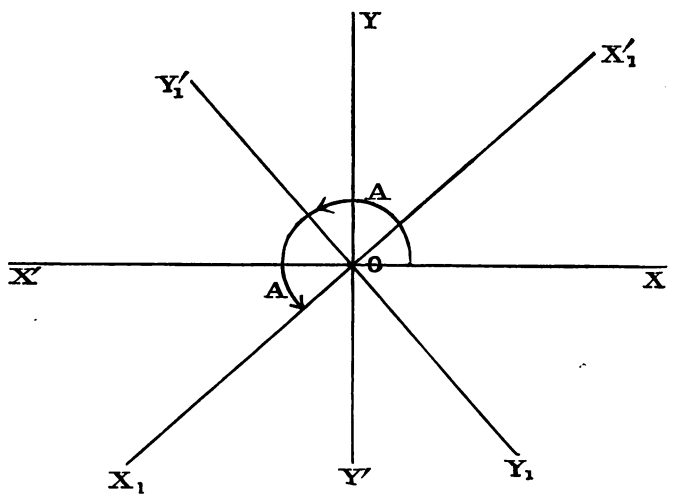
$$U_p = \lambda^p \frac{d^p}{d\lambda^p}F(\lambda) = u^p F^{(p)}(u).$$

Other examples might be written down.

A general proof of the Addition Theorems in Trigonometry.

By W. E. Philip, M.A.

It is a very convenient method to begin the study of Trigonometry with one or two lessons on Coordinates and the Coordinate Diagram. This leads to a general definition of the Sine and Cosine of any angle which the beginner has little difficulty in comprehending. With the help of the Coordinate Diagram the theorems on projection which are of so much importance can be proved with great precision. It is in such a course that the present proof of the Addition Theorems might find a place.



Let X'OX, Y'OY be the axes of coordinates. Let them be rotated round O in the positive direction through an angle A which may be of any magnitude.

The new position of the x-axis is represented on the diagram by OX_1 which thus makes an angle A with OX.

The new position of OY is OY_1 which makes an angle $90° + A$ with OX.

Any point P on the plane can now be specified in two ways:

either (1) by its coordinates x, y referred to the old axes

or (2) by its coordinates ξ, η „ „ „ new axes.

If M_1 be any point on OX_1 the projections of OM_1 on the x and y axes are $OM_1\cos A$, $OM_1\sin A$.

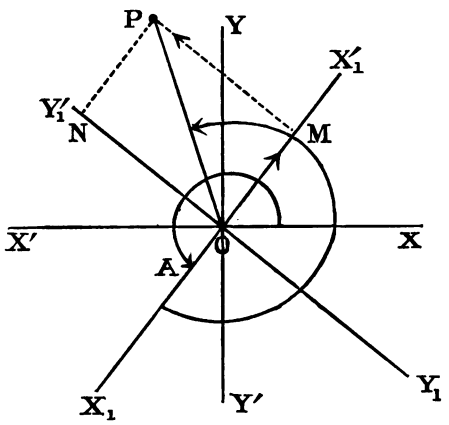
If M_1' be any point on OX_1' the projections of OM_1' on the x and y axes are $-OM_1'\cos A$, $-OM_1'\sin A$.

Thus we may make the following general statement :—

If M be any point on the line X_1OX_1' specified by the coordinate ξ, the projections of OM on the x and y axes are $\xi\cos A$, $\xi\sin A$.

Again if N be any point on the line Y_1OY_1' specified by the coordinate η, the projections of ON on the x and y axes are

$$\eta\cos(A+90°), \quad \eta\sin(A+90°).$$



Formulae for the sine and cosine of $(A+B)$.

Let the revolving line of length r start from OX and trace out an angle A. It will then coincide with OX_1. Let it further revolve through an angle B and take up the position OP indicated in the diagram, the coordinates of P referred to the new axes being ξ, η. The line OP now makes an angle $(A+B)$ with OX.

If we consider the diagram X_1OX_1', Y_1OY_1' we have an angle B traced out by the revolving line starting from the initial position OX_1. Thus

$$\cos B = \frac{\xi}{r}, \quad \sin B = \frac{\eta}{r},$$

by the definition of the sine and cosine of an angle.

Now draw PM, PN perpendiculars to X_1OX_1', Y_1OY_1'.

The journey from O to P can be performed in two ways:

(1) along OP,

(2) along OM and then along MP.

By the theorem on projections we have

projection of OP = projection of OM + projection of MP

= projection of OM + projection of ON.

Take these on the x-axis:

$$r\cos(A+B) = \xi\cos A + \eta\cos(A+90°)$$
$$= r\cos B\cos A + r\sin B\cos(A+90°);$$
$$\therefore\ \cos(A+B) = \cos A\cos B - \sin A\sin B,$$

since $\cos(A+90°) = -\sin A$.

Again take projections on the y-axis:

$$r\sin(A+B) = \xi\sin A + \eta\sin(A+90°)$$
$$= r\cos B\sin A + r\sin B\sin(A+90°);$$
$$\therefore\ \sin(A+B) = \sin A\cos B + \cos A\sin B,$$

since $\sin(A+90°) = +\cos A$.

To find formulae for the sine and cosine of $(A-C)$.

The position OP of the revolving line may be attained in the following way:

First by a revolution through an angle $+A$ and then a further revolution through an angle $-C$. Hence by considering the coordinate diagram given by the axes X_1OX_1', Y_1OY_1' we have

$$\xi = r\cos(-C),\quad \eta = r\sin(-C)\quad \text{by definition.}$$

The angle which OP now makes with OX is $(A-C)$, and thus by the same theorems as before

$$r\cos(A-C) = \xi\cos A + \eta\cos(A+90°)$$
$$= r\cos(-C)\cos A + r\sin(-C)\cos(A+90°);$$
$$\therefore\ \cos(A-C) = \cos A\cos C + \sin A\sin C,$$

since $\cos(-C) = +\cos C$, $\sin(-C) = -\sin C$.

Again $r\sin(A-C) = \xi\sin A + \eta\sin(A+90°)$;

$$\therefore\ \sin(A-C) = \sin A\cos C - \cos A\sin C.$$

APPENDIX.

The following paper on *Proportion* in its original form was remitted to a Committee of the Edinburgh Mathematical Society for consideration and report. After discussion in Committee it was reported to the Society, and at the meeting held on 12th January 1900 the following motion was unanimously adopted:—

> "The Edinburgh Mathematical Society resolves that Professor GIBSON's Paper on Proportion be printed in its *Proceedings*, and recommends it to Mathematical Teachers as a suitable and sufficient substitute for Euclid's Fifth Book."

Proportion: A Substitute for the Fifth Book of Euclid's "Elements."

By Prof. GEORGE A. GIBSON.

INTRODUCTION.

The following proposed substitute for the fifth book of Euclid's *Elements* has been drawn up with the object of filling a gap that undoubtedly exists in the current methods of teaching elementary geometry. It is a matter of common knowledge that Euclid's fifth book is rarely read in schools, and, in spite of its intrinsic excellence, it is not at all likely to be reintroduced into elementary teaching.

The chief reason—and it is a strong reason—for retaining Euclid's treatment of proportion is that it puts commensurable and incommensurable magnitudes on the same level; but this reason loses all its force if, as is usually the case, only the definitions and not the propositions of the fifth book are read. Besides, even when the fifth book has been studied, it is necessary to go further and to show that ratio as defined by Euclid is a quantity that can be used in calculations like ordinary numbers before the properties of ratios established in the fifth and sixth books of the *Elements* can logically be applied as is universally done in such a subject as trigonometry. In other words, since Euclid does not define a ratio as a number, proof ought to be given that it can be compared with numbers and that it has the same laws of combination as numbers before it can be legitimately subjected to the operations of algebra as its use in trigonometry, and applied geometry in general, requires. Such proof is rarely, if ever, attempted, not to say effected.

In whatever way a ratio may be defined to begin with, any treatment that is to be of the slightest use in practice must at some stage or other show that a ratio is a quantity subject to the laws of algebra. The common observation that the fifth book of Euclid is not geometrical, tells in favour of an arithmetical theory of proportion rather than against it, for it shows that Euclid could not develop

his system on geometrical grounds alone. It would, indeed, be strange if a system of geometry which, like Euclid's, is essentially metrical, were independent of considerations of number. The difficulties attaching to the ratios of incommensurable magnitudes are identical with those of irrational numbers, and are not at all geometrical in their nature. The objection sometimes urged against an arithmetical treatment of proportion, that geometrical theorems should be established by geometrical methods alone, is without force as an argument in favour of Euclid's treatment, since his theorems on proportion are quite independent of geometry. The fifth book of Euclid is in reality a magnificent treatise on abstract number, not at all a treatise on geometry; but for that very reason it is quite unsuitable, as experience has proved, for elementary teaching.

The most natural method seems to be to begin with commensurable magnitudes, just as arithmetic begins with integers. When the pupil has in this way acquired some familiarity with the properties of ratios and their use in geometry, he can then have the difficulties of incommensurable magnitudes brought before him. In whatever way the subject be treated, the difficulties of the ratios of incommensurable magnitudes, like those of irrational numbers, are considerable, and a full discussion would probably be beyond the capacity of the average school-boy. In the following articles an effort has been made to render the conception of the ratio of two incommensurable magnitudes as clear as possible, and to emphasise the connection with the fundamental conception of a limit, but there is no attempt to prove the combining laws of such ratios. There is the less need for such proof, as in the forthcoming second edition of the second volume of Professor Chrystal's *Algebra* there is a thorough discussion of the irrational number. It is hoped that sufficient has been done to give the pupil clear and accurate ideas of the nature of an irrational number or of the ratio of two incommensurable magnitudes, and thus to prepare him for the fuller treatment that will be found in such an exposition as that referred to, as well as for the applications of the method of exhaustions or of limits in his more advanced studies.

It is therefore strongly recommended that the pupil should for a first reading confine himself to §§ 1–10, omitting §§ 11–16, and going on to the proof of the theorems in proportion that are required in geometry.

Some propositions from Euclid's sixth book are appended, to show how they may be proved when Euclid's definition of proportion is replaced by that given in this paper.

Ratio and Proportion of Magnitudes.

1. If A and B denote two like magnitudes,* that is, two magnitudes of the same kind, *e.g.*, two straight lines or two rectangles, the sum of A and B will be denoted by $A+B$ and the difference by $A-B$, when A is greater than B, but by $B-A$ when A is less than B.

The sum of n magnitudes, each of which is equal to A, will be denoted by nA, and the magnitude nA will be said to contain A n times, or n times exactly.

2. *Definition.* If one magnitude contain another magnitude a certain number of times exactly, the greater is called a *multiple* of the less and the less is called a *sub-multiple* or a *measure* or an *aliquot part* of the greater.

If A contains B n times, A is called the nth multiple of B, and B the nth submultiple or the nth part of A.

These relations may be expressed by the equations

$$A = nB, \quad B = \frac{A}{n}.$$

3. If A, B be like magnitudes and m, n be integers, it follows from the first principles of arithmetic that

$$m(A \pm B) = mA \pm mB$$

$$(m \pm n)A = mA \pm nA$$

$$n \,.\, mA = nmA = mnA = m \,.\, nA$$

$$\frac{A \pm B}{n} = \frac{A}{n} \pm \frac{B}{n}.$$

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* In the fifth book of Euclid, magnitudes are usually represented by straight lines, and this procedure is very convenient, as it helps to make the somewhat abstract reasoning more definite. In class teaching, of course, the conclusions stated for integers m, n, etc., should be illustrated by the use of definite numbers, 2, 3, etc. On the comparison of magnitudes in general, see § 15.

Again, m times the nth part of A, that is $m\frac{A}{n}$, is the same magnitude as the nth part of m times A, that is $\frac{mA}{n}$. Each of these expressions $m\frac{A}{n}$ and $\frac{mA}{n}$ may therefore be represented by $\frac{m}{n}A$, so that the expression pA has a perfectly definite meaning even when p is a fraction. The arithmetic of fractions then shows that the above equations hold true even when m, n are fractional numbers.

4. *Definition.* When each of two like magnitudes is a multiple of a third magnitude, the third magnitude is called a *common measure* of the two magnitudes, and the two magnitudes are said to be *commensurable;* if the two magnitudes have no common measure, they are said to be *incommensurable.*

It will be proved in §11 that there are incommensurable magnitudes; §§5–10 deal with commensurable magnitudes only.

5. *Theorem.* If M be a common measure of A and B, every measure of M will also be a common measure of A and B. For a magnitude that contains M a certain number of times will contain $\frac{1}{2}$M, $\frac{1}{3}$M, etc., twice, thrice, etc., that number of times.

This theorem has the following converse: if G be the greatest common measure of A and B, every common measure of A and B will also be a measure of G.

For, let $A = aG$, $B = bG$ and let M be any other common measure of A and B.

Since G is the greatest common measure of A and B, the numbers a and b will be prime to each other; for if a and b had a common measure c, then cG would be a measure both of A and of B, and therefore G would not be the greatest common measure.

Now suppose $A = mM$, $B = nM$.

Therefore, $mM = aG$, $nM = bG$

or $M = \frac{a}{m}G$, $M = \frac{b}{n}G$.

Hence $\frac{a}{m} = \frac{b}{n}$ or $\frac{m}{n} = \frac{a}{b}$.

But $\frac{a}{b}$ is a fraction in its lowest terms, and therefore $m = ra$, $n = rb$ where r is the greatest common measure of m and n.

Hence $M = \frac{a}{m}G = \frac{1}{r}G$, and therefore M is a measure of G.

6. From what has just been said about common measures it appears that any two commensurable magnitudes have an unlimited number of common measures; the smaller the measure is, the greater is the number that expresses the multiple which the magnitude is of its measure. But it has been proved that if M be any common measure of A and B, and if $A = mM$, $B = nM$, the fraction $\frac{m}{n}$ has the same value whatever common measure M be taken, being equal to the fraction $\frac{a}{b}$ where a, b are the number of times A, B respectively contain their greatest common measure.

Definition. If A and B be two like magnitudes having a common measure M, so that

$$A = mM, \quad B = nM, \quad \text{and therefore} \quad A = \frac{m}{n}B$$

the ratio of A to B is defined to be the fraction $\frac{m}{n}$.*

If A be greater than $\frac{m}{n}B$, the ratio of A to B is defined to be greater than the fraction $\frac{m}{n}$; and if less, less.

From what has just been stated the ratio of A to B is a number which is independent of the size of the common measure M.

It is stated in the definition that A and B are magnitudes of the same kind, and when the ratio of two magnitudes is spoken of, it is always to be understood that they are of the same kind.

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* Instead of saying that "the ratio is equal to $\frac{m}{n}$," we sometimes say that "the ratio is measured by the fraction $\frac{m}{n}$"; but as a mathematical quantity it is the numerical value of the ratio which we always have in view.

Again, in arithmetic the ratio of two integers m and n is defined to be the fraction $\frac{m}{n}$, so that the definition of ratio given above is equivalent to the following:—"When $A = \frac{m}{n}B$, the ratio of A to B is equal to the ratio of m to n.

The ratio of A to B is often expressed by the notation A : B. A and B are called the *terms* of the ratio; the term which comes first is called the *antecedent* and the other the *consequent.*

NOTE.—If A be equal to B, the ratio A : B is unity, and the ratio is in this case sometimes spoken of as *a ratio of equality.* If A be greater than B, the ratio A : B is an improper fraction and the ratio is then spoken of as a *ratio of greater inequality* or a *ratio of majority*, while if A be less than B, the ratio A : B is a proper fraction and is then spoken of as a *ratio of less inequality* or a *ratio of minority.*

7. *Definition.* If A and B be two like magnitudes, and if C and D be two other like magnitudes, though not necessarily of the same kind as A and B, the four magnitudes A, B, C, D are defined to be *proportionals* or to be *in proportion* when the ratio of A to B is equal to the ratio of C to D.

The proportion is often expressed by the notation

$$A : B = C : D \text{ *}$$

or, in words, "A is to B as C is to D."

The magnitudes A, B, C, D are called the *terms* of the proportion; the first and fourth terms are called the *extremes*, the second and third the *means.* Thus A and D are the extremes, B and C the means.

When four magnitudes are proportionals, the first and third terms are said to be *homologous* to each other, and the second and fourth are also said to be *homologous* to each other. Instead of *homologous terms* the expression *corresponding terms* may be used, and the first and third may then be said to correspond.

When four magnitudes are proportionals, the fourth magnitude is sometimes called *the fourth proportional* to the other three.

It is obvious that C is greater than, equal to, or less than D according as A is greater than, equal to, or less than B.

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* The notation A : B : : C : D is sometimes used instead of A : B = C : D.

8. *Definition.* If there be any number of like magnitudes greater than two, of which the first has to the second the same ratio that the second has to the third, and the second to the third the same ratio that the third has to the fourth, and so on, the magnitudes are said to be *continual proportionals* or *in continued proportion.*

When three magnitudes are in continued proportion, the second is said to be *the mean proportional* between the other two, and the third is said to be *the third proportional* to the other two.

Magnitudes in continued proportion are sometimes said to be in *geometrical progression*, and when there are three magnitudes in continued proportion the second is then called *the geometric mean* between the other two.

9. *Definition.* The ratio of B to A is defined to be *the reciprocal* or *the inverse* of the ratio of A to B.

From the definition of proportion it follows that

$$\text{if} \quad A : B = C : D$$
$$\text{then} \quad B : A = D : C$$

for the ratios $A : B$ and $C : D$ are each equal to the fraction $\frac{m}{n}$, and their reciprocals are each equal to the fraction $\frac{n}{m}$.

Again, the first of these two proportions may evidently be written in the form

$$C : D = A : B$$

and the second in the form

$$D : C = B : A$$

so that the truth of any one of these proportions implies the truth of the other three.

10. When the ratio of A to B is greater than the ratio of C to D, the relation between the magnitudes may be expressed by the notation

$$A : B > C : D,$$

while if the ratio of A to B is less than that of C to D, the relation may be expressed by the notation

$$A : B < C : D.$$

It is clear that

$$\text{if} \quad A : B > C : D$$
$$\text{then} \quad B : A < D : C.$$

Again, the first of these inequalities may be written in the form

$$C : D < A : B$$

and the second in the form

$$D : C > B : A$$

so that from any one of them the other three follow.

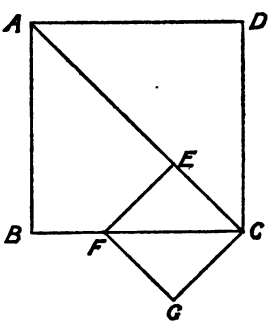
11. The magnitudes which have been considered up to this point have been supposed to be commensurable; it will now be proved that there are pairs of incommensurable magnitudes.

In proving that incommensurable magnitudes exist, and more generally in seeking to define the ratio of such magnitudes, the following axiom or postulate, called the axiom of Archimedes, is assumed:—

Given two unequal magnitudes of the same kind, there always exists a multiple of the less that is greater than the greater magnitude, or—which amounts to the same thing—there always exists a submultiple of the greater that is less than the less of the two given magnitudes.

To prove that the diagonal and the side of a square are incommensurable.

From the diagonal AC, cut off AE equal to AB or BC, and draw EF perpendicular to AC to meet BC at F.



Then $BF = EF = EC$.

Hence EF, EC are two sides of a square EFGC, of which FC is a diagonal, and the side EC consequently less than half BC.

Now $AC - BC = EC$ (1)

$BC - EC = FC$ (2)

Equation (1) shows that every common measure of AC and BC is a measure of EC, while equation (2) shows that every common measure of BC and EC is a measure of FC.

Hence every common measure of AC and BC is also a common measure of FC and EC, and EC has been shown to be less than half BC; that is, every common measure of the diagonal and the side of a given square is also a common measure of the diagonal and the side of another square, the side of which is less than half the side of the given square.

The theorem can now be applied to the square EFGC, and so on indefinitely. If the theorem be applied n times in all, the conclusion is that every common measure of the diagonal and the side of a given square is also a common measure of the diagonal and the side of another square, the side of which is less than $1/2^n$ of the side of the given square.

If there be a common measure of AC and BC, let it be the line X. Now by the axiom of Archimedes we can take n so large that $BC/2^n$ shall be less than X. But the common measure sought for must be a measure of a line that is less than $BC/2^n$, no matter how large n may be. X therefore can not be a common measure of AB and BC since X is greater than $BC/2^n$. Hence the diagonal and the side of a square can have no common measure.

It has therefore been proved that there is one pair of incommensurable magnitudes;* as a matter of fact, incommensurable magnitudes are not exceptions of rare occurrence.

12. The general method of treating incommensurable magnitudes may be illustrated by considering the diagonal and the side of a square.

Denote AC by D and BC by S; then clearly

$$D > 1\ S \quad \text{but} \quad D < 2\ S.$$

Divide BC into 10 equal parts; then BC^2 contains 100 squares, the side of each being S/10, while AC^2 contains more than 196 but less than 225 such squares. Hence

$$D > 1{\cdot}4\ S \quad \text{but} \quad D < 1{\cdot}5\ S.$$

Proceeding in this way, it may be shown that

$$D > 1{\cdot}41\ S \quad \text{but} \quad D < 1{\cdot}42\ S$$
$$D > 1{\cdot}414\ S \quad \text{but} \quad D < 1{\cdot}415\ S$$
$$D > 1{\cdot}4142\ S \quad \text{but} \quad D < 1{\cdot}4143\ S$$

and so on.

In this way two sets of approximations to D are obtained, the one set being in defect, the other in excess; these may be called the lower and the upper sets respectively.

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* As another instance of a pair of incommensurable lines, the segments of a straight line divided in medial section may be taken. See Mackay's *Euclid*, Book II., Prop. 11, Cor. 1.

It is important to notice that the difference between corresponding members of the two sets gets less and less as the work proceeds, and further if we assume the axiom of Archimedes we can go so far as to make the difference between D and any approximation less than any given line. For if the given line be X, we can choose n so large that $S/10^n$ shall be less than X; if the work be carried out to n decimals, then the difference between the lower and the upper approximations is $S/10^n$, and therefore the difference between D and either of them less than $S/10^n$ and *a fortiori* less than X.

Hence although we can find no rational number a such that $D = aS$, we can find two rational numbers a, b such that

$$D > aS \quad \text{but} \quad D < bS$$

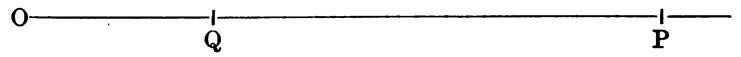
and at the same time $(b - a)S$ less than any given line,

In arithmetic the irrational number $\sqrt{2}$ is defined as the number which is greater than any of the numbers belonging to the lower set of approximations to D, namely, 1, 1.4, 1.41, etc., and less than any of the numbers belonging to the upper set, 2, 1.5, 1.42, etc.

Hence we may write $D = \sqrt{2}S$.

This symbol $\sqrt{2}$ is subject to the laws of algebra, and is, for that reason if for no other, called a *number;* it is, however, an *irrational* number, and the numbers 1, 1.4, 1.41, etc., are called rational approximations to it.

13. We may now take the general case of two incommensurable straight lines; let these be OP, OQ and suppose OP greater than OQ



We form the two sets of approximations to OP by considering (1) how often OP contains OQ, with a remainder less than OQ, (2) OQ/10 with remainder less than OQ/10, (3) $OQ/10^2$ with remainder less than $OQ/10^2$, and so on.

The lower set of approximations will consist of the multiples of OQ, OQ/10, $OQ/10^2$, etc., thus obtained, while the upper set will be these multiples increased by unity. Denote the lower set of approximations by a_1OQ, a_2OQ, a_3OQ, etc., and the corresponding

members of the upper set by b_1OQ, b_2OQ, b_3OQ, etc.; then we get the following scheme. The first approximation gives

$$OP > a_1OQ \quad \text{but} \quad OP < b_1OQ, \quad (b_1 - a_1)OQ = OQ$$

the second gives

$$OP > a_2OQ \quad \text{but} \quad OP < b_2OQ, \quad (b_2 - a_2)OQ = OQ/10$$

the third gives

$$OP > a_3OQ \quad \text{but} \quad OP < b_3OQ, \quad (b_3 - a_3)OQ = OQ/10^2$$

and generally, the nth gives

$$OP > a_nOQ \quad \text{but} \quad OP < b_nOQ, \quad (b_n - a_n)OQ = OQ/10^{n-1}.$$

Now, by the axiom of Archimedes, we can choose n so large that $OQ/10^{n-1}$ shall be less than any given line. Hence we can find a rational number a_n or b_n such that OP shall differ from a_nOQ or b_nOQ by less than any given line.

But further, suppose the two sets of numbers a_1, a_2, etc., b_1, b_2, etc., are all known; then there cannot be two different lines which are both greater than every one of the approximations in defect and both less than every one of those in excess. For, if possible, let OP, OP′ be two such lines, and let OP′ be greater than OP, say $OP' - OP = D$.

We can choose n so large that $OQ/10^{n-1}$ shall be less than D; but $$OP' < b_nOQ, \quad OP > a_nOQ.$$

Therefore, $$OP' - OP < (b_n - a_n)OQ.$$

But $$(b_n - a_n)OQ = OQ/10^{n-1} < D.$$

Hence $OP' - OP$ is less than D, which contradicts the supposition that $OP' - OP = D$. In the same way it may be shown that OP is not greater than OP′. Hence $OP' = OP$.

Thus the two sets of approximations determine the line OP uniquely when OQ is given. We therefore introduce a symbol α and write $OP = \alpha OQ$, where α is defined by this property that it is greater than every one of the numbers a_1, a_2, etc., of the lower set and less than every one b_1, b_2, etc., of the upper set. α is an irrational number, and the a's and b's are rational approximations to it.

These rational approximations are expressed as decimals; but this has been done for convenience merely. The reasoning would hold equally well if, for example, instead of OQ, OQ/10, OQ/10², etc., we had chosen OQ, OQ/2, OQ/3, etc. The essential point is that

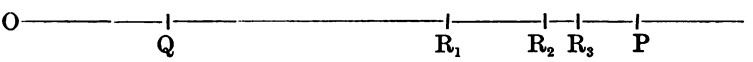
(1) the a's should give approximations in defect and the b's approximations in excess, and (2) that by taking n sufficiently large we should be able to make $(b_n - a_n)OQ$ less than any given line, or $(b_n - a_n)$ less than any given fraction.

The symbol α has been called an irrational number, and it is shown in treatises on algebra * that it may be used according to the same rules that apply to rational numbers. In practice, approximations are always sufficient, and in place of the irrational number a rational approximation may be substituted according to the degree of accuracy required, so that even though the full theory of irrational numbers be not presupposed, there is, by the use of the approximations, an amount of control that is sufficient for all practical purposes.

14. There is another way of looking at the matter that is instructive. Consider the series of lines

$$a_1OQ, \quad a_2OQ, \ldots\ldots \qquad a_nOQ, \ldots\ldots$$

If we measure off these lines from O in the direction OP, we get for their second extremities, say, the points R_1, R_2, etc. As we pass from R_1 to R_2, then to R_3, and so on, we get nearer and nearer to P,



and though we never in this way quite reach P, we can come nearer to P than by any given distance; P is a *boundary* or a *limit* to our advance. Hence it is usual to speak of OP as the *limit* of the variable line a_nOQ, the variation being made by supposing n to increase indefinitely; and in the same way α is called the limit of the numbers a_n. In this case the limit is greater than each of the approximations: but clearly we might equally well regard OP as the limit of the lines b_nOQ where OP is less than each of the approximations.

It is also clear from this way of considering the matter that the knowledge of *one only* of the two sets of approximations would be sufficient to determine OP. In the general theory of limits it is not necessary that the approximations should be either always in defect

* See Chrystal's *Algebra*, Vol. II. (second edition).

or always in excess; the essential thing is that we should be able to find n so that for that value and all greater values the difference between OP and a_nOQ shall be less than any given line.

We may express the above result in the notation

$$\mathrm{OP} = \lim_{n=\infty} (a_n \mathrm{OQ}), \qquad \alpha = \lim_{n=\infty} a_n ;$$

in words, OP is the limit for n increasing indefinitely of a_nOQ.

15. The considerations that have been applied to the comparison of straight lines hold for many of the other magnitudes of elementary geometry. Rectilineal areas can be supposed converted into rectangles of equal altitude, and one of these can be divided into equal areas by first dividing the base into equal parts. The comparison is then of exactly the same nature as in the case of straight lines. Angles, and arcs, and sectors of equal circles can be compared by superposition, but we must assume the possibility of subdivision into equal parts. Certain solids—for example, prisms—can be treated in a similar way; but, as a rule, outside this range, definitions are necessary to bring the magnitudes within the scope of mathematical treatment. Thus, before an arc of a circle can be compared with a straight line, some definition is required of the phrase "arc of a circle." In such cases recourse is usually had to the consideration of limits; for example, the circumference of a circle is considered as the limit of the perimeter of an inscribed (or circumscribed) polygon when the number of its sides is increased indefinitely, the length of each side at the same time diminishing indefinitely.

16. We may now define the ratio of two incommensurable magnitudes as follows:—

If A, B are two like incommensurable magnitudes, and if B be divided into any number n of equal parts of which A contains more than m but less than $m+1$, so that

$$\mathrm{A} > \frac{m}{n}\mathrm{B} \quad \text{but} \quad \mathrm{A} < \frac{m+1}{n}\mathrm{B}$$

then the ratio of A to B is defined to be the irrational number which is greater than every number of the set m/n and less than every number of the set $(m+1)/n$.

It is clear that the ratio of two incommensurable magnitudes can not be equal to the ratio of two commensurable magnitudes, for that would mean that an irrational number can be equal to a rational number.

The definition of proportion is the same as before, but it can also be put into a form that is sometimes convenient in practice, namely,

If A, B be two like magnitudes, and if C, D be two other like magnitudes, though not necessarily of the same kind as A, B, then A, B, C, D will form a proportion if when

$$A > \frac{m}{n}B \quad \text{but} \quad < \frac{m+1}{n}B$$

at the same time

$$C > \frac{m}{n}D \quad \text{but} \quad < \frac{m+1}{n}D$$

and that for every value of n.

That this definition is equivalent to the first follows from the fact that the ratios of A to B and C to D are both equal to the irrational number determined by the sets m/n and $(m+1)/n$.

The proofs given of theorems in proportion for commensurable magnitudes hold equally for incommensurables, for the proofs depend on the hypothesis that the ratio of two magnitudes may be written in the form $A : B = k : 1$
where k is a symbol that is subject to the rules of algebra.

17. The theorems in proportion that are required in elementary geometry will now be stated. The proofs are only given in a few cases, as they all run on the same lines and are very similar to those found in text-books of algebra. Capital letters are used throughout to denote magnitudes, and small letters to denote numbers.

THEOREM 1

$$pA : pB = A : B$$

where p is any number.

THEOREM 2

If $A : B = C : D$, then $pA : qB = pC : qD$

where p, q are any numbers.

For, let each of the equal ratios A : B, C : D be equal to the ratio $k : 1$; then

$$A = kB, \quad C = kD.$$

$$\therefore \quad pA : qB = pkB : qB = pk : q$$

$$pC : qD = pkD : qD = pk :$$

$$\therefore \quad pA : qB = pC : qD$$

since each of these ratios is equal to $pk : q$.

THEOREM 3

If $A = B$, then $A : C = B : C$.

Conversely, if $A : C = B : C$, then $A = B$.

THEOREM 4

If $A > B$, then $A : C > B : C$

Conversely, if $A : C > B : C$, then $A > B$.

If $A < B$, then $A : C < B : C$

Conversely, if $A : C < B : C$, then $A < B$.

THEOREM 5

If $A : B = C : D$, then $B : A = D : C$

Proved in § 9.

This inference is referred to as *inversely*, or, *by inversion.*

THEOREM 6

If the magnitudes A, B, C, D be all of the same kind, and if

$$A : B = C : D, \text{ then } A : C = B : D.$$

For let each of the equal ratios A : B, C : D be equal to the ratio $k : 1$; then $\quad A = kB, \quad C = kD$

$$\therefore \quad A : C = kB : kD = B : D \quad \text{by Th. 1.}$$

This inference is referred to as *alternately*, or, *by alternation.*

THEOREM 7

If $A : B = C : D$, then $A + B : B = C + D : D$.

This inference is referred to as *by addition*, or, *by composition*.

THEOREM 8

If $A : B = C : D$, then $A - B : B = C - D : D$, when $A > B$, and therefore $C > D$, but $B - A : B = D - C : D$, when $A < B$, and therefore $C < D$.

This inference is referred to as *by subtraction*, or *by division*.

THEOREM 9

If $A : B = C : D$, then $A + B : A - B = C + D : C - D$,

or $A + B : B - A = C + D : D - C$,

according as A is greater or less than B, and therefore C greater or less than D.

This inference is referred to as *by addition and subtraction*, or *by composition and division*.

For the proof, take the case of Theorem 9 when A is greater than B. As before, let each of the equal ratios $A : B$, $C : D$ be equal to the ratio $k : 1$; then

$$A = kB ; \quad \therefore \quad A + B = (k+1)B, \quad A - B = (k-1)B$$

$$\therefore \quad A + B : A - B = (k+1)B : (k-1)B = k+1 : k-1.$$

Similarly, $C + D : C - D = k+1 : k-1$

$$\therefore \quad A + B : A - B = C + D : C - D.$$

THEOREM 10

If the magnitudes A, B, C, D be all of the same kind, and if $A : B = C : D$, then each ratio is equal to the ratio $A + C : B + D$ or to the ratio $A - C : B - D$, when $A > C$ and therefore $B > D$, but to the ratio $C - A : D - B$ when $A < C$ and therefore $B < D$.

Or, *in words*, as one antecedent is to its consequent, so is the sum of the antecedents to the sum of the consequents, or the difference of the antecedents to the difference of the consequents.

The proof is obvious, since in the same notation as before.

$$A + C = kB + kD = k(B + D)$$

and therefore

$$A + C : B + D = k : 1 = A : B = C : D.$$

This theorem is a particular case of a more general theorem, which may be stated as follows and may be proved in the same way :—

If the magnitudes A, B, C, D...R, S be all of the same kind, and if $A : B = C : D = \ldots = R : S$, then each ratio is equal to the ratio $\quad mA \pm nC \pm \ldots \pm pR : mB \pm nD \pm \ldots \pm pS$

where m, n,...p are any integers. It is, of course, to be remembered that in taking the difference of magnitudes, negative magnitudes are not considered.

THEOREM 11

If the magnitudes A, B, C, D, E, F,...R, S be all of the same kind, and if the ratios A : B, C : D, E : F,...R : S be not all equal, then the ratio

$$A + C + E + \ldots + R : B + D + F + \ldots + S$$

is less than the greatest but greater than the least of the ratios

$$A : B,\ C : D,\ E : F,\ldots R : S.$$

For suppose A : B to be the greatest and R : S the least of these ratios, and let R : S be equal to the ratio of $k : 1$; then

$$A : B > k : 1 \quad \therefore \quad A > kB,$$

Similarly, $\quad C > kD,\ E > kF,\ldots R = kS$

$$\therefore \quad A + C + E + \ldots + R > k(B + D + F + \ldots + S)$$

$$\therefore \quad A + C + E + \ldots + R : B + D + F + \ldots + S > k : 1 > R : S.$$

In the same way it may be proved that

$$A + C + E + \ldots + R : B + D + F + \ldots + S < A : B.$$

There is an obvious extension of this theorem corresponding to the general case of Theorem 10, but in this case only sums and not differences are to be taken.

18. Since a ratio is a number, ratios may be multiplied or divided. It is to be observed, however, that it is the *ratios* that are multiplied or divided; we may multiply the *ratio* of A to B by the *ratio* of C to D, but we do not multiply the *magnitude* A by the *magnitude* C, or the *magnitude* B by the *magnitude* D.

The product of the ratios A : B and C : D will be represented by the notation

$$(A : B) \times (C : D) \quad \text{or} \quad (A : B)\ (C : D).$$

The quotient of the ratio A : B by the ratio C : D may be represented by the notation

$$(A : B)/(C : D) \quad \text{or} \quad (A : B) \div (C : D) \quad \text{or} \quad (A : B) : (C : D)$$

or, in short, in any of the ways for expressing a quotient in arithmetic.

Euclid employs the phrase "to compound ratios" in the same sense as "to multiply ratios." Hence the

Definition. To compound two or more ratios means to take the product of the ratios.

It follows at once from the definition that if there be any number of like magnitudes A, B, C, D, the ratio A : D is the ratio compounded of the ratios A : B, B : C, C : D.

For, let $A : B = k : 1$, $B : C = l : 1$, $C : D = m : 1$;

then $A = kB$, $B = lC$, $C = mD$

$\therefore\ A = kB = klC = klmD$

$\therefore\ A : D = klm : 1.$

But $(A : B)(B : C)(C : D) = (k : 1)(l : 1)(m : 1) = klm : 1$

$\therefore\ A : D = (A : B)(B : C)(C : D).$

Again, since a ratio is merely a number, if

$$P : Q = A : B,\ R : S = B : C,\ T : V = C : D,$$

then $(P : Q)(R : S)(T : V) = (A : B)(B : C)(C : D)$

and $\therefore\ (P : Q)(R : S)(T : V) = A : D.$

The ratio compounded of two or more ratios can therefore be expressed as the ratio of two magnitudes.

Euclid employs special names to denote the ratio compounded of two or more equal ratios.

Definitions. The ratio compounded of two equal ratios is called the *duplicate* of either; the ratio compounded of three equal ratios is called the *triplicate* of any one of them; and so on.

The duplicate of A : B may be written (A : B)(A : B) or $(A : B)^2$; the triplicate may be written $(A : B)^3$; and so on.

In other words, the duplicate of a ratio is the square of it; the triplicate of a ratio is the cube of it; and so on. It would be much better to discard the phrases *duplicate ratio*, *triplicate ratio*, and the terminology of *compound ratios* altogether; the use of *duplicate ratio* is specially objectionable.

In geometrical work it is often convenient to express the square (or the duplicate) of the ratio of two magnitudes as the simple ratio of the two magnitudes. This may be done as follows :—

Let A, B be the two given magnitudes; find C the third proportional to A, B so that

$$A : B = B : C.$$

Then A : C is the square (or duplicate) of A : B; for

$$A : C = (A : B)(B : C) = (A : B)(A : B) = (A : B)^2.$$

In the same way, to find the cube (or the triplicate) of A : B, find C, D so that

$$A : B = B : C \quad \text{and} \quad B : C = C : D$$

then $$A : D = (A : B)^3.$$

Higher powers of a ratio may be dealt with in the same way.

19. The following theorems involving the products of ratios are often required.

THEOREM 12.

If two ratios are equal, their squares (duplicates), their cubes (triplicates), etc., are equal.

Conversely, if the squares (duplicates), the cubes (triplicates), etc., of two ratios are equal, the ratios themselves are equal.

The proof is mere arithmetic since negative and imaginary numbers are not in question; for let $x : 1$, $y : 1$ be the two ratios, then if $x : 1 = y : 1$, it follows that $(x : 1)^n = (y : 1)^n$ when n is any positive integer.

Conversely, if $(x : 1)^n = (y : 1)^n$, it follows that $x : 1 = y : 1$.

THEOREM 13.

If A, B, C be three like magnitudes, and if P, Q, R be three other like magnitudes, though not necessarily of the same kind as A, B, C, such that

$$A : B = P : Q$$
$$\text{and } B : C = Q : R$$

then $$A : C = P : R.$$

For
$$\begin{aligned} A : C &= (A : B)(B : C) \\ &= (P : Q)(Q : R) \\ &= P : R. \end{aligned}$$

More generally, if there be two sets of magnitudes

A, B, C,...H, K, L and P, Q, R,...X, Y, Z

such that

$$A : B = P : Q$$
$$B : C = Q : R$$
$$\cdots\cdots \quad \cdots\cdots$$
$$H : K = X : Y$$
$$K : L = Y : Z$$

then $$A : L = P : Z$$

The proof is as before,

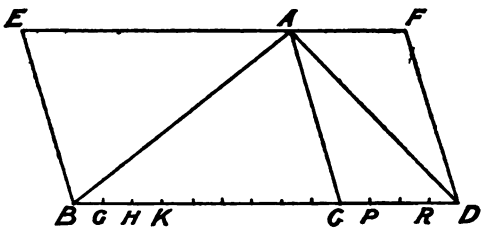
This inference is referred to as "by equality."

20. When Euclid's treatment of proportion is not adopted, the proofs of certain propositions in the sixth book require to be altered. The alteration is absolutely necessary only in the cases of the first and thirty-third propositions, but alternative proofs are given for one or two others. It seems more natural to prove such fundamental propositions as the sixteenth and nineteenth without the use of reciprocal proportionals. In fact, the only use to which Euclid puts the conception of reciprocal proportionals in plane geometry is to establish Propositions 14 and 15 as stepping-stones to Propositions 16, 17, and 19. Although much use was made in the ancient geometry of reciprocal proportionals, there would be little if any loss in discarding reciprocal proportionals from elementary geometry. The demands of examinations rather than the necessities of geometry seem, however, to compel the retention of Propositions 14 and 15, and the definition of reciprocal proportionals.

EUCLID, VI. 1.

Triangles and parallelograms of the same altitude are to one another as their bases.

Let the triangles ABC, ACD and the parallelograms EC, CF have the same altitude, namely, the perpendicular from A to BD or BD produced:



it is required to prove that

triangle ABC : triangle ACD = BC : CD

and parallelogram EC : parallelogram CF = BC : CD.

First, let the bases BC, CD be commensurable, and let BG be a common measure of BC, CD.

Mark off on BC, CD the lines GH, HK....., CP,.....RD each equal to BG and suppose A joined to the points G, H,..... R.

Because BG, GH, HK, ... CP, ... RD are all equal, therefore,

the triangles ABG, AGH, AHK, ... ACP, ... ARD are all equal.

Hence,

if BC contain BG m times, $\triangle ABC$ will contain $\triangle ABG$ m times, and if CD contain BG n times, $\triangle ACD$ will contain $\triangle ABG$ n times. Therefore, the two ratios BC : CD and the $\triangle ABC : \triangle ACD$ are each equal to the fraction $\frac{m}{n}$, and are therefore equal to one another.

Hence, $\triangle ABC : \triangle ACD = BC : CD.$

Second, let the bases BC, CD be incommensurable.

Suppose that CD contains BG n times; then BC will not contain BG any number of times exactly. Let BC be greater than mBG but less than $(m+1)$BG.

As before, $\triangle ACD$ will contain $\triangle ABG$ n times and $\triangle ABC$ will be greater than m times but less than $(m+1)$ times $\triangle ABG$.

$\therefore$ the two ratios BC : CD and $\triangle ABC : \triangle ACD$ are each greater than the fraction $\frac{m}{n}$ but less than the fraction $\frac{m+1}{n}$, and that

no matter how great n may be. Hence the two ratios are equal to the same irrational number, and are therefore equal to one another.

Therefore, $\triangle ABC : \triangle ACD = BC : CD.$

Again, the parallelograms EC, CF are double of the triangles ABC, ACD respectively

$\therefore$ parallelogram EC : parallelogram CF = BC : CD.

Cor. 1. Triangles and parallelograms that have equal altitudes are to one another as their bases.

Cor. 2. Triangles and parallelograms that have equal bases are to one another as their altitudes.

Cor. 3. If BC, CD be any two lines and XY any other line,

$$BC : CD = BC \cdot XY : CD \cdot XY$$

for BC . XY, CD . XY are rectangles of the same altitude XY.

[In the same way Euclid VI., 33, may be proved.]

EUCLID VI. 16, 17.

If four straight lines be proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means.

Conversely: If the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportional.

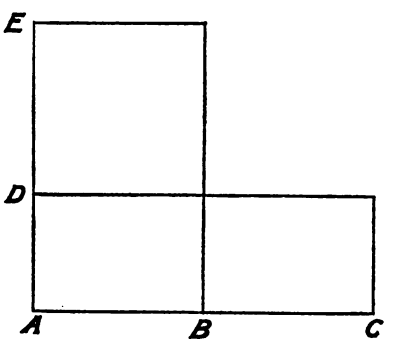
(1) Let the four straight lines AB, AC, AD, AE be proportional, so that

$$AB : AC = AD : AE :$$

it is required to prove

$$AB \cdot AE = AC \cdot AD.$$

Let AE be perpendicular to AC and complete the rectangles CD, BE.



$$AB : AC = \text{rectangle BD} : \text{rectangle CD}$$

and $$AD : AE = \text{rectangle BD} : \text{rectangle BE}.$$

But $AB : AC = AD : AE$

therefore, rectangle BD : rectangle CD = rectangle BD : rectangle BE

therefore, rectangle BE = rectangle CD

that is $AB . AE = AC . AD.$

(2) Let $AB . AE = AC . AD$:

it is required to prove $AB : AC = AD : AE.$

Make the same construction as before.

$AB : AC$ = rectangle BD : rectangle CD

and $AD : AE$ = rectangle BD : rectangle BE.

But rectangle BE = rectangle CD

therefore, $AB : AC = AD : AE.$

Cor. If AC be equal to AD, the proposition may be stated : –

If three straight lines be proportional, the rectangle contained by the extremes is equal to the square on the mean.

Conversely : if the rectangle contained by the extremes be equal to the square on the mean, the three straight lines are proportional.

Euclid VI. 19.

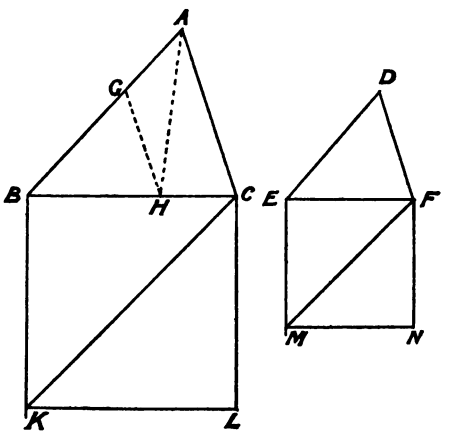
Similar triangles are to one another in the duplicate ratio of their homologous sides.

Let ABC, DEF be two similar triangles, having the angles at B, C equal to the angles at E, F respectively :

it is required to prove that

$\triangle ABC : \triangle DEF = (BC : EF)^2$

From BA, BC or from BA, BC produced, cut off BG, BH equal to ED, EF respectively, and join GH, AH.



Then the triangle GBH is congruent with the triangle DEF, and therefore GH is parallel to AC.

$$\triangle ABC : \triangle ABH = BC : BH$$

and $$\triangle ABH : \triangle GBH = BA : BG$$
$$= BC : BH$$

therefore, $(\triangle ABC : \triangle ABH)(\triangle ABH : \triangle GBH) = (BC : BH)(BC : BH)$

that is, $$\triangle ABC : \triangle GBH = (BC : BH)^2$$

or, $$\triangle ABC : \triangle DEF = (BC : EF)^2.$$

Cor. If on BC, EF the squres BL, EN be drawn, the squares are double of the triangles KBC, MEF respectively, and these triangles are similar.

$$\therefore \quad BC^2 : EF^2 = \triangle KBC : \triangle MEF$$
$$= (BC : EF)^2.$$

Hence, the duplicate ratio of two lines is equal to the ratio of the squares on the lines, and therefore similar triangles are to one another as the squares on their homologous sides.

Euclid VI., 22.

See figures in *Mackay's Euclid*, or, *Todhunter's Euclid.*

(1) $$KAB : LCD = (AB : CD)^2$$
$$MF : NH = (EF : GH)^2$$
But $$AB : CD = EF : GH$$
$\therefore$ $$(AB : CD)^2 = (EF : GH)^2$$
$\therefore$ $$KAB : LCD = MF : NH$$

(2) $$KAB : LCD = (AB : CD)^2$$
$$MF : NH = (EF : GH)^2$$
But $$KAB : LCD = MF : NH$$
$\therefore$ $$(AB : CD)^2 = (EF : GH)^2$$
$\therefore$ $$AB : CD = EF : GH$$

EUCLID VI. 23.

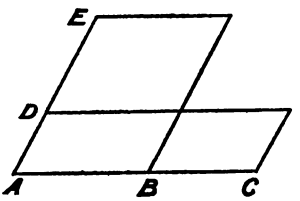
Mutually equiangular parallelograms have to one another the ratio which is compounded of the ratios of their sides.

Let parallelogram BE be equiangular to parallelogram CD, and let

$$\angle EAB = \angle DAC;$$

to prove

$$\|^{m}BE : \|^{m}CD = (AB : AC)(AE : AD).$$



Because $\|^{m}BE : \|^{m}BD = AE : AD$

and $\|^{m}BD : \|^{m}CD = AB : AC$

therefore $(\|^{m}BE : \|^{m}BD)(\|^{m}BD : \|^{m}CD) = (AE : AD)(AB : AC)$

But $(\|^{m}BE : \|^{m}BD)(\|^{m}BD : \|^{m}CD) = \|^{m}BE : \|^{m}CD$

therefore $\|^{m}BE : \|^{m}CD = (AB : AC)(AE : AD).$

Cor. $\|^{m}BE : \|^{m}CD = AB . AE : AC . AD$

Suppose the sides AB, AE and AC, AD to remain constant and the angle A to vary; then the areas of the parallelograms BE and CD will vary, but the ratio of their areas will not vary, since this ratio is always equal to (AB : AC)(AE : AD).

If the angle A become a right angle, the parallelograms BE and CD will become the rectangles AB . AE and AC . AD.

Hence $AB . AE : AC . AD = (AB : AC)(AE : AD)$

and therefore $\|^{m}BE : \|^{m}CD = AB . AE : AC . AD.$

This corollary establishes the theorem that "the ratio compounded of the ratios of two pairs of lines is equal to the ratio of the rectangle contained by the antecedents to the rectangle contained by the consequents."

Cor. 2. If BE and CD be joined, the triangles EAB, DAC are halves of the parallelograms BE, CD. Hence,

Two triangles which have one angle of the one equal to one angle of the other have to each other the same ratio as the rectangles contained by the sides about the equal angles.

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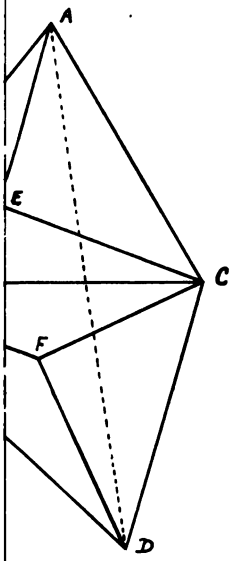

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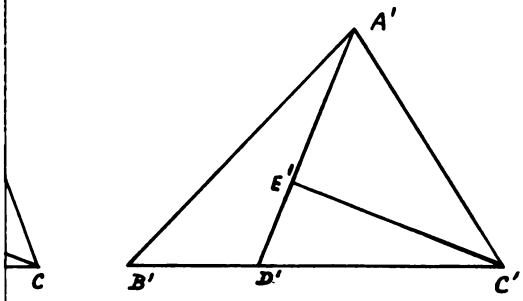

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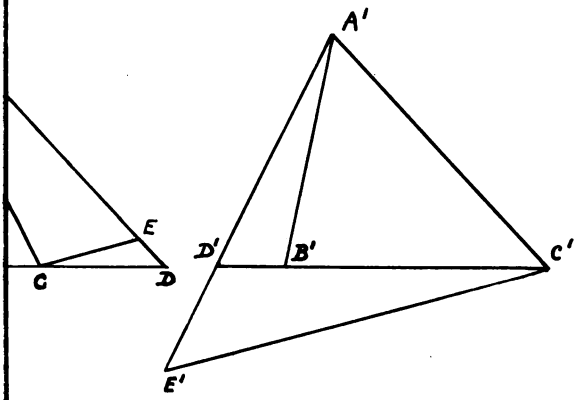

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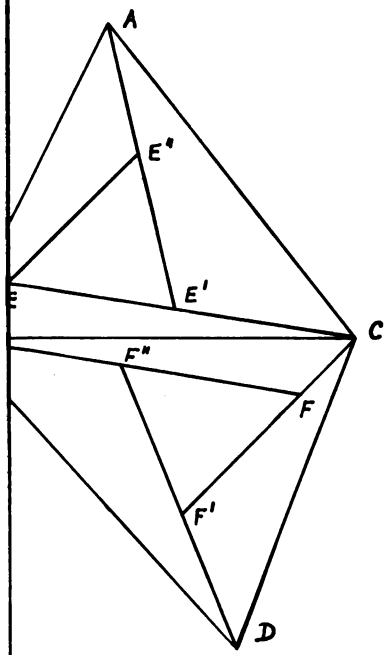

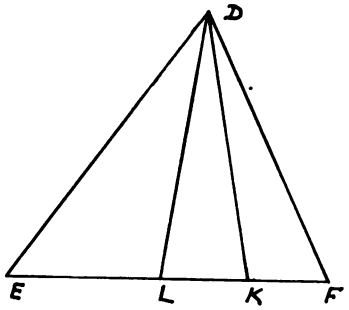

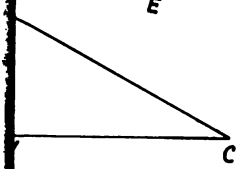

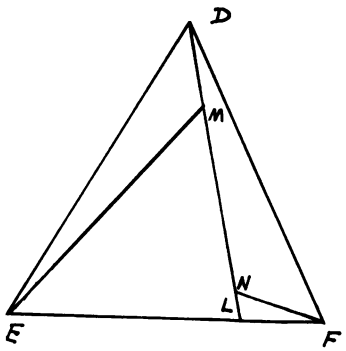

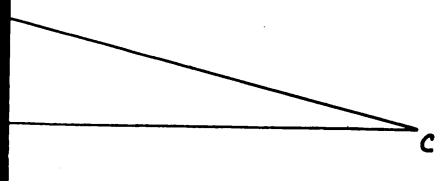

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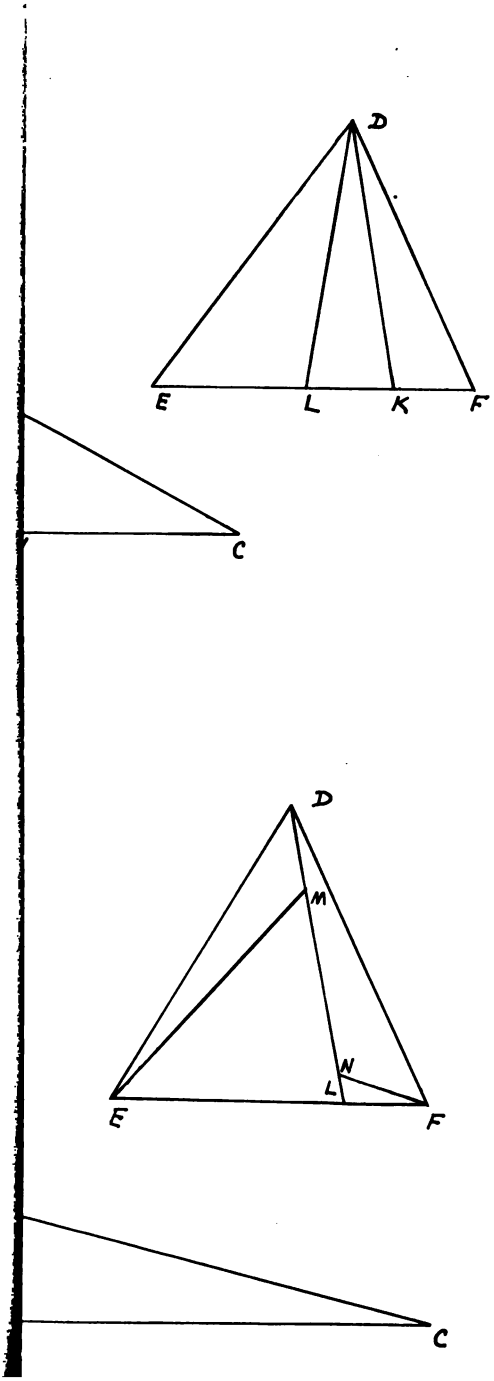
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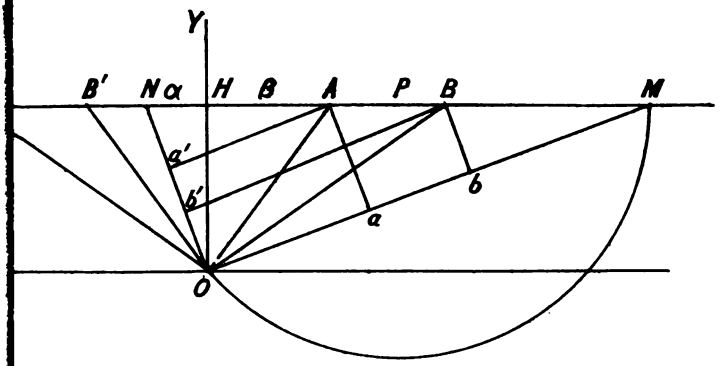

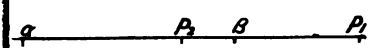

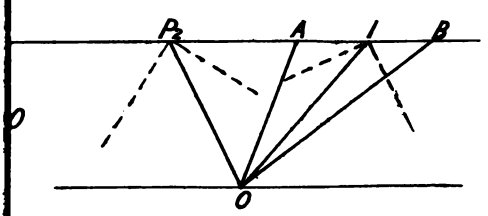

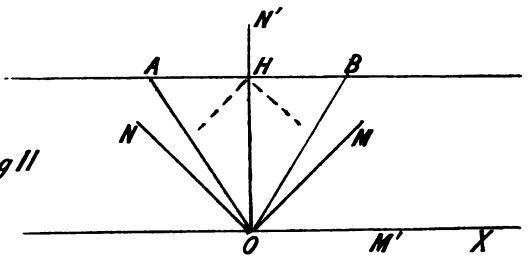

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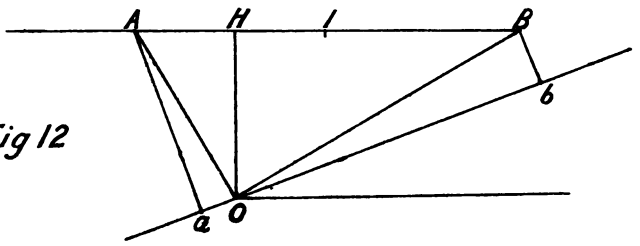

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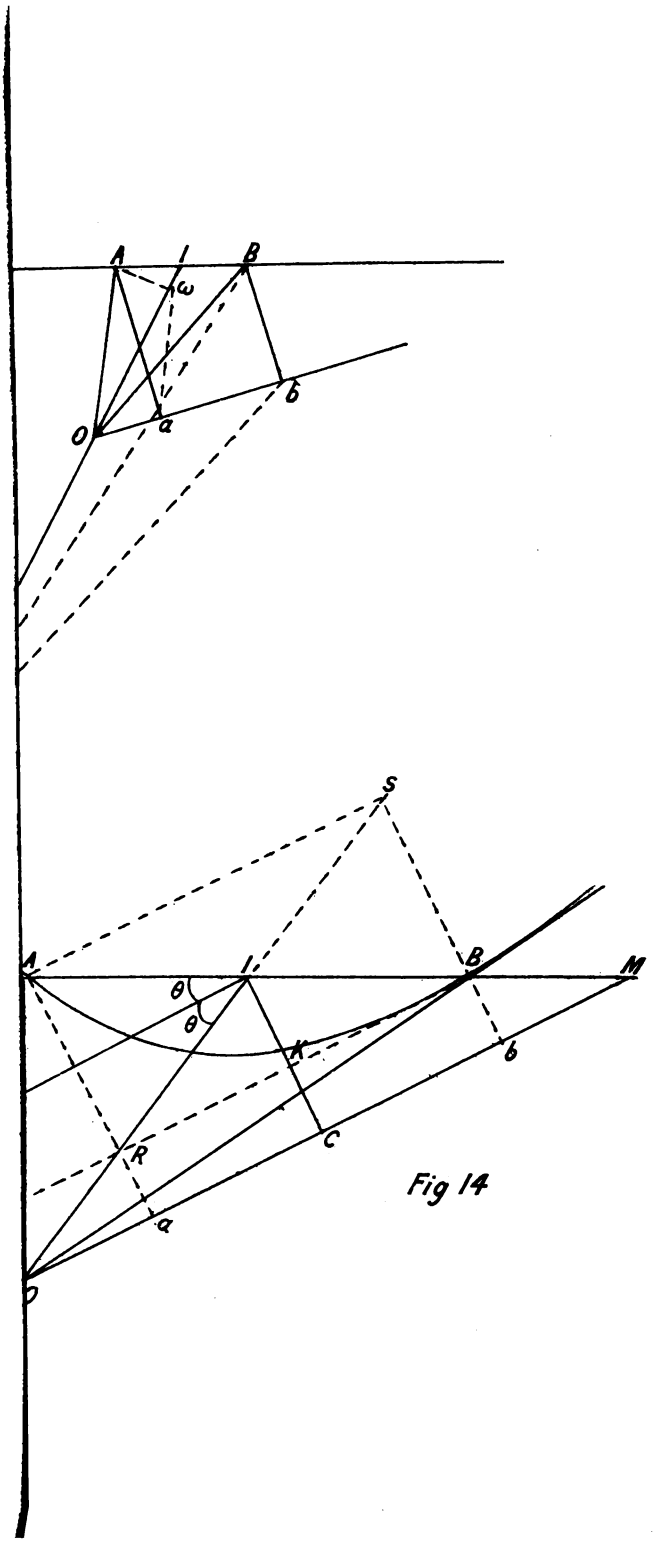

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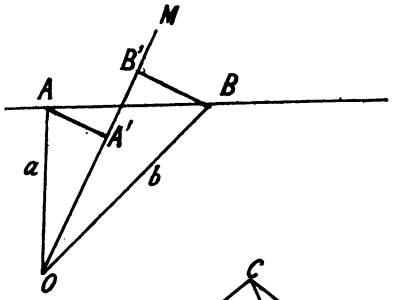

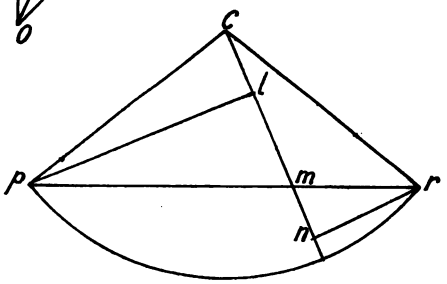

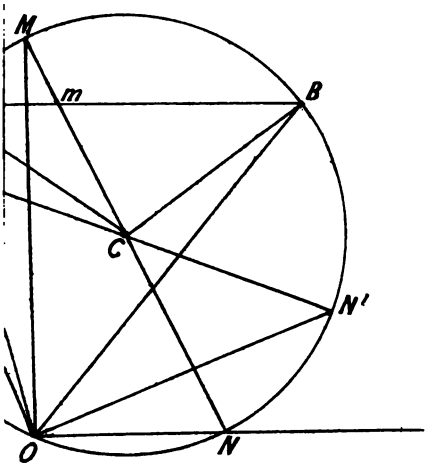

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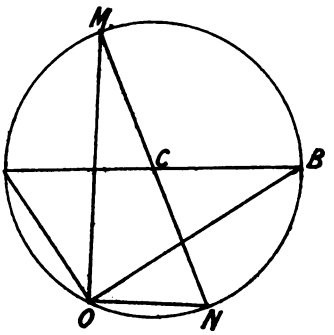

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